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Supersymmetric mechanics with Kähler phase space

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Contents

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Chapter 1

Introduction

Kähler manifolds are widely used in numerous sections of modern theoretical and mathematical physics [1],[3],[10],[33]. It is mostly considered as a configuration space of various systems. Here Kähler manifolds are considered as a phase space of some (super)integrable systems, which lead us to an interesting and elegant description of integrability, rooted in the underlying geometry of the phase space.

One of the advantages of considering integrable systems on them is that the Kähler structure of the phase spaces enables the use of the geometric quantization method [11],[12], [13], [14]. The number of known nontrivial (super)integrable systems featuring a Kähler phase space is quite limited, and their examination remains at the periphery of integrable systems theory. This is particularly surprising, given that the quantization of systems with a Kähler phase space has been a focal point in modern geometry ever since the inception of geometric quantization. A notable integrable model with a Kähler phase space that is currently under extensive investigation is the (compactified) Ruijesnaars-Schneider model [5] ,[6], [7], [8]. However, even this system is primarily studied in canonical coordinates. On the other hand, establishing a connection between existing integrable systems and their constants of motion with the isometries of a Kähler manifold viewed as a phase space can be useful in comprehending the system's geometry. It's an important step towards quantization in non-canonical coordinates.

This thesis is devoted to the study of classical mechanical (super)integrable systems having

Kähler phase spaces and a construction of their supersymmetric extensions. As demonstrated later in this work, the methods that are used herein can be extremely useful for supersymmetrization of a given model. The phase superspaces of such a supersymmetric extensions are Kähler supermanifolds, meaning, a Kähler manifold equipped with Grassmann anticommuting coordinates also ([10],[16]).

The thesis comprises four chapters (excluding the current introductory chapter). The second chapter is dedicated to a renowned classical system known as the Euler top. It is considered as a one-dimensional system with Kähler phase space, specifically $\mathbb{CP}^1$, and its supersymmetrization in that context. It is based on [18]. In the third chapter, we study compact and non-compact complex projective spaces, along with their symmetries. Focusing on the non-compact case, we regard them as phase spaces of N-dimensional conformal mechanics, N-dimensional oscillator, and N-dimensional Coulomb system. Their integrability properties are studied from the geometrical point of view. This material can be found in [17]. In the fourth chapter, we examine the possibilities of supersymmetrization within this formalism and consider the options for supersymmetrization for the examples from the previous chapter [19]. Finally, the fifth chapter is devoted to discussion of the main results and possible future developments of the ideas explored in this thesis.

This chapter focuses on the basic introductory information about Kähler manifolds, Hamiltonian formalism, and supersymmetric mechanics. These concepts are widely used in current work.

Section 1.1 is dedicated to Kähler manifolds and explains how they serve as phase spaces for different systems. It covers the basics and their application in this context.

In *Section 1.2* we talk about Hamiltonian formalism in general, as well as, consider Hamiltonian and super-Hamiltonian reduction technique.

Section 1.3 focuses on integrability .

And finally, *Section 1.4* is devoted to supersymmetric mechanics.

1.1 KÄHLER MANIFOLDS

Kähler manifolds are Hermitian manifolds that possess a symplectic structure, satisfying specific compatibility conditions with Riemann (and/or complex) structures [1], [3]. These manifolds commonly appear as configuration spaces for particles and fields in theoretical physics. While they can be considered as phase spaces for Hamiltonian systems, this usage is limited to certain physical problems, such as various versions of the Hall effect and tops [64]. The number of known nontrivial (super)integrable systems with K"ahler phase spaces is restricted, and their study is on the margin of the theory of integrable systems.

An exceptional integrable model with a Kähler phase space extensively studied nowadays is the compactified Ruijesnaars-Schneider model with an excluded center of mass, whose phase space is the complex projective space $\mathbb{CP}^N$ [5]. However, even in these investigations, canonical coordinates are used.

On the other hand, there are indications that Kähler phase spaces can be useful for studying conventional Hamiltonian systems, particularly those formulated on the cotangent bundle of Riemann manifolds. For instance, one-dimensional conformal mechanics formulated in terms of Lobachevsky plane, serving as a noncompact complex projective plane, plays the role of phase space [9]. This elegant description allows for the immediate construction of its $\mathcal{N} = 2M$ superconformal extension associated with $su(1,1|M)$ superalgebra. In recent paper [17], a similar formulation of some higher-dimensional systems was presented in terms of $su(1,N)$-symmetric Kähler phase space, considered as a noncompact version of complex projective space.

Relating angular coordinates and momenta with the action-angle variables (e.g. [15]) of the angular part of the integrable conformal mechanics, all symmetries of generic superintegrable conformal-mechanical systems are described in terms of the powers of the $su(1, N)$ isometry generators. The maximally superintegrable generalizations of the Euclidean oscillator/Coulomb systems express all symmetries of these systems via $su(1, N)$ isometry generators. However, supersymmetrization aspects of that system were not considered there. In a subsequent paper [19], we have studied supergeneralizations of those systems, we have constructed their $\mathcal{N}$-extended superconformal extensions , as done in [9] for the one-dimensional case. These results are presented in Chapter 4.

According to Darboux's theorem, any symplectic structure can locally be presented in the canonical form corresponding to canonical Poisson brackets. Furthermore, any cotangent bundle of a Riemann manifold can be equipped with the globally defined canonical symplectic structure. Hence, for the Hamiltonian description of systems of particles moving on the Riemann space, we can restrict ourselves to the canonical symplectic structure (and canonical Poisson brackets). Non-canonical Poisson brackets are usually used for the description of more sophisticated systems, such as various modifications of tops, (iso)spin dynamics, etc.

As it was mentioned above, Kähler manifolds have three mutually compatible structures, namely complex structure, Riemannian structure and symplectic structure. Kähler manifold is a particular case of the general Hermitian manifold ($g_{a\bar{b}} dz^a d\bar{z}^b$). For any Hermitian metric one can define a 2-form

$$\omega = \imath g_{a\bar{b}} dz^a \wedge d\bar{z}^b \qquad (1.1)$$

This 2-form is called a fundamental form. Hermitian manifold is called *Kähler* if this 2-form is symplectic (closed and non-degenerate). This condition puts a strong limitation, which allows us to express a Kähler metric as the second derivative of a function known as the Kähler potential.

$$g_{a\bar{b}} = \frac{\partial^2 K(z, \bar{z})}{\partial z^a \partial \bar{z}^b} \qquad (1.2)$$

It is worth to mention that it is defined up to a holomorphic or antiholomorphic function:$K(z, \bar{z}) \rightarrow$

$K(z,\bar{z})+U(z)+\bar{U}(\bar{z})$.

Symplectic structure of Kähler manifolds allows as naturally equip it with Poisson brackets.

$$\{f,g\}_0 = ig^{a\bar{b}}\Big(\frac{\partial f}{\partial z^a}\frac{\partial g}{\partial \bar{z}^b}-\frac{\partial g}{\partial z^a}\frac{\partial f}{\partial \bar{z}^b}\Big), \quad g^{a\bar{b}}g_{\bar{b}c}=\delta^a_c. \tag{1.3}$$

Since the symplectic structure relates functions (Hamiltonian) and vector fields (Hamiltonian vector fields), we can introduce functions, which generate Killing vector fields.

$$\mathbf{V}_\mu = \{h_\mu, \}_0 = V^a_\mu\frac{\partial}{\partial z^a}+\bar{V}^{\bar{a}}_\mu\frac{\partial}{\partial \bar{z}^a}, \qquad V^a_\mu = -ig^{a\bar{b}}\partial_{\bar{b}}h_\mu \tag{1.4}$$

Such functions are called Killing potentials. By employing the Killing Equations, limitations on Killing potentials can be deduced. These potentials must have real values and must satisfy the following equation.

$$\frac{\partial^2 h_\mu}{\partial z^a\partial z^b}-\Gamma^c_{ab}\frac{\partial h_\mu}{\partial z^c}=0 \tag{1.5}$$

These functions are extremely useful for studying systems on Kähler manifolds in presence of a constant magnetic field. Since any 2-form is closed in two (real) dimensions, a one-dimensional orientable complex manifold (Riemann surface) can always be equipped with a Kähler structure. Many components of the Christoffel symbols and Riemann tensor vanish.

$$\Gamma^a_{bc}=g^{a\bar{d}}g_{b\bar{d},c}, \qquad R^a_{bc\bar{d}}=-(\Gamma^a_{bc})_{,\bar{d}}. \tag{1.6}$$

In this thesis some (super)integrable models on maximally symmetric Kähler manifolds are discussed. Namely, in the first chapter an integrable system Euler top is approached as a one dimensional system possessing $\mathbb{CP}^1$ as a phase space. Latter on, N dimensional compact and non-compact projective spaces are involved as phase spaces of some model. And namely the non-compact projective space, denoted here as $\widetilde{\mathbb{CP}}^N$, is studied in details, as phase space of N-dimensional conformal mechanics and its deformations, specifically Oscillator-like and Coulomb-like models.

1.1.1 $\mathbb{C}^{N+1}$ AND $\mathbb{C}^{N.1}$ AS A KÄHLER MANIFOLD

From now on, as we start more concrete consideration and for being more precise let us denote coordinates on Euclidean spaces (both, compact and non-compact) by $u, \bar{u}$, leaving $z, \bar{z}$ for the projective spaces. As well as, use i, j, k for the indices, actually, for the same reason. Certainly, it is very artifical to consider $N+1$-dimensional Euclidean space instead of N-dimensional. We do so just for a complete analogy with the non-compact case (where we do not consider $\mathbb{C}^{N-1.1}$), and it is also useful for the next section where we discus compact and non-compact projective spaces, which are the main objects of study in this thesis, and we take them to be N-dimensional.

The metric of the $N+1$-dimensional complex Euclidean space is well known

$$ds^2 = du d\bar{u}, \qquad g_{i\bar{j}} = \delta_{i\bar{j}}. \tag{1.7}$$

Here and throughout the remainder of this text, when we express a product without explicit indices, it implies the presence of dummy indices and the performance of a subsequent summation:

$$du d\bar{u} \equiv du_i d\bar{u}_i \equiv \sum_i du_i d\bar{u}_i, \quad z\bar{z} \equiv z_a \bar{z}_a \equiv \sum_a z_a \bar{z}_a, \quad \text{etc.} \tag{1.8}$$

It is easy to note that the Kähler potential and the symplectic structure are as follows

$$K(u, \bar{u}) = u\bar{u}, \qquad \omega = -\imath du \wedge d\bar{u}, \qquad \{u^i, \bar{u}^j\}_0 = \imath \delta^{i\bar{j}}, \tag{1.9}$$

Such a Kähler potential leads to the metric above. All the components of Christoffel symbols and Riemann tensor vanish. Finally we present the results for Killing potentials and corresponding Killing vector fields.

$$\begin{aligned}
&h_{i\bar{j}} = \bar{u}^i u^j, \qquad \mathbf{V}_{i\bar{j}} = -\imath(u^i \partial_i + \bar{u}^i \partial_{\bar{j}}) \\
&h_i^+ = \bar{u}^i, \qquad \mathbf{V}_i^- = -\imath \partial_i, \\
&h_i^- = u^i, \qquad \mathbf{V}_i^+ = -\imath \partial_{\bar{i}}.
\end{aligned} \tag{1.10}$$

$V_{i\bar{j}}$ vector fields generate rotations, while V_i^- and V_i^+ are the generators of translation. Although $h_{i\bar{j}}$, h_i^+ and h_i^- are not real, one can take real combinations using these functions. The number of real Killing potentials is $(N+1)(2(N+1)+1)$, so as is mentioned $\mathbb{C}^{N+1}$ is maximally symmetric space.

The metric of $N+1$-dimensional complex pseudo-Euclidean space $\mathbb{C}^{N.1}$ is equally well known

$$ds^2 = du_0 d\bar{u}_0 - du_a d\bar{u}_a, \qquad g_{i\bar{j}} = \gamma_{i\bar{j}}. \tag{1.11}$$

where $\gamma = \mathrm{diag}(1, -1, \dots, -1)$ and indices $a, b, ... = 1, ..., N$.

Kähler potential and associated Poisson brackets are given by

$$K(u, \bar{u}) = u_0 \bar{u}_0 - u_a \bar{u}_a, \qquad \{u_i, \bar{u}_j\} = \imath \gamma_{i,\bar{j}}. \tag{1.12}$$

The Killing potentials and associated vector fields are are given by (1.10).

1.1.2 $\mathbb{CP}^N$ AND $\widetilde{\mathbb{CP}}^N$ AS A KÄHLER MANIFOLD

The N-dimensional complex projective space is a space of complex rays in the $(N+1)$-dimensional complex Euclidian space $(\mathbb{C}^{N+1}, \sum_{i=0}^N du^i d\bar{u}^i)$, with u^i being homogeneous coordinates of the complex projective space. Equivalently, it can be defined as the quotient $\mathbb{S}^{2N+1}/U(1)$, where $\mathbb{S}^{2N+1}$ is the $(2N+1)$-dimensional sphere embedded in $\mathbb{C}^{N+1}$ by the constraint $\sum_{i=1}^N u^i \bar{u}^i = 1$. One can solve the latter by introducing locally "inhomogeneous" coordinates $z^a_{(i)}$

$$z^a_{(i)} = \frac{u^a}{u^i}, \qquad \text{with} \quad a \neq i, u^i \neq 0. \tag{1.13}$$

Hence, the full complex projective space can be covered by $N+1$ charts marked by the indices $i = 0, \dots, N$, with the following transition functions on the intersection of i-th and j-th charts:

$$z^a_{(i)} = \frac{z^a_{(j)}}{z^i_{(j)}}. \qquad (1.14)$$

Let us endow $\mathbb{C}^{N+1}$ with the canonical Poisson brackets $\{u^i, \bar{u}^j\} = \imath\delta^{i\bar{j}}$, and define, with respect to them, the $u(N+1)$ algebra formed by the generators

$$h_{i\bar{j}} = \bar{u}^i u^j \,. \qquad (1.15)$$

Reducing the manifold $\mathbb{C}^{N+1}$ by the action of the $U(1)$ group with the generator $J = u^i\bar{u}^i$, we arrive at the $SU(N+1)$-invariant Kähler structure defined by the Fubini-Study metrics ([38], [72])

$$g_{a\bar{b}}dz^a d\bar{z}^b = \frac{\partial^2 \log(1+z\bar{z})}{\partial z^a \partial \bar{z}^b} dz^a d\bar{z}^b = \frac{dzd\bar{z}}{1+z\bar{z}} - \frac{(\bar{z}dz)(zd\bar{z})}{(1+z\bar{z})^2}, \quad K = \log(1+z\bar{z}). \qquad (1.16)$$

This metrics is obviously invariant under the passing from one chart to another. Hence, we can omit the indices marking charts and assume, without loss of generality, that we are dealing with 0-th chart, so that the indices a, b, c run from 1 to N.

Being Kähler manifold, the complex projective space is equipped with the Poisson brackets $\{z^a, \bar{z}^b\} = \imath g^{a\bar{b}}$, where $g^{a\bar{b}} = (1+z\bar{z})(\delta^{a\bar{b}} + z^a\bar{z}^b)$ is the inverse Fubini-Study metrics. The $su(N+1)$ isometry of $\mathbb{CP}^N$ is generated by the holomorphic Hamiltonian vector fields defined as the following momentum maps (Killing potentials).

$$h_{a\bar{b}} = \frac{\bar{z}^a z^b}{1+z\bar{z}}, \qquad h^-_a = \frac{\bar{z}^a}{1+z\bar{z}}, \qquad h^+_a = \frac{z^a}{1+z\bar{z}}. \qquad (1.17)$$

Like for the Euclidean case the number of independent Killing vector fields indicates that this space is again maximally superintegrable. Finally we can compute the components of Christoffel symbol and Riemann tensor.

$$\Gamma^a_{bc} = -\frac{\delta^a_b \bar{z}^c + \delta^a_c \bar{z}^b}{1+z\bar{z}}, \quad R_{a\bar{b}c\bar{d}} = g_{a\bar{b}}g_{c\bar{d}} + g_{c\bar{b}}g_{a\bar{d}}, \qquad (1.18)$$

An analogues formulas can be written down for the non compact case. As it was mentioned, to get to the non-compact complex projective space we start from $N+1$ dimensional pseudo-Euclidean space $\mathbb{C}^{N.1}$ with Poisson brackets $\{u_i, \bar{u}_j\} = \imath\gamma_{i,\bar{j}}$, where γ is as defined in previous section.

Reducing $\mathbb{C}^{N.1}$ by the action of $U(1)$ generator $J = u^0\bar{u}^0 - u^a\bar{u}^a$ we arrive at the $SU(1.N)$ invariant Kähler structure

$$g_{a\bar{b}}dz^a d\bar{z}^b = \frac{\partial^2 \log(1 - z\bar{z})}{\partial z^a \partial \bar{z}^b} dz^a d\bar{z}^b = \frac{dz d\bar{z}}{1 - z\bar{z}} + \frac{(\bar{z}dz)(zd\bar{z})}{(1 - z\bar{z})^2}, \quad K = \log(1 - z\bar{z}). \tag{1.19}$$

This manifold is equipped with the following Poisson brackets

$$\{z^a, \bar{z}^b\} = \frac{\imath}{g} g^{a\bar{b}}, \tag{1.20}$$

where $g^{a\bar{b}} = (1 - z\bar{z})(\delta^{a\bar{b}} - z^a\bar{z}^b)$ is the inverse metric. The isometry algebra of $\widetilde{\mathbb{CP}}^N$ is $su(1.N)$, and it is generated by the holomorphic Hamiltonian vector fields defined by the following Killing potentials

$$h_{a\bar{b}} = \frac{\bar{z}^a z^b}{1 - z\bar{z}}, \qquad h_a^- = \frac{\bar{z}^a}{1 - z\bar{z}}, \qquad h_a^+ = \frac{z^a}{1 - z\bar{z}}. \tag{1.21}$$

Again, as in compact case the number of Killing potentials shows that this a maximally symmetric space. Here are the components of Christoffel symbols and Reimann tensor

$$\Gamma^a_{bc} = \frac{\delta^a_b \bar{z}^c + \delta^a_c \bar{z}^b}{1 - z\bar{z}}, \quad R_{a\bar{b}c\bar{d}} = g_{a\bar{b}}g_{c\bar{d}} + g_{c\bar{b}}g_{a\bar{d}}, \tag{1.22}$$

1.2 HAMILTONIAN FORMALISM

Below, we recall some very basic facts about Hamiltonian formalism, which can be found in many textbooks on classical mechanics or, to align with the spirit of this thesis, in [10]. Non-degenerate Poisson brackets of a Hamiltonian system are locally defined by

$$\{f,g\} = \frac{\partial f}{\partial x^i}\omega^{ij}(x)\frac{\partial g}{x^j}, \qquad \det\omega \neq 0. \tag{1.23}$$

This brackets obey the following conditions, which are known as antisymmetricity condition and Jacobi identity

$$\{f,g\} = -\{g,f\}, \tag{1.24}$$

$$\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0, \tag{1.25}$$

or equivalently

$$\omega_{ij} = -\omega_{ji}, \tag{1.26}$$

$$\omega_{ij,k} + \omega_{ki,j} + \omega_{jk,i} = 0. \tag{1.27}$$

Due to the nondegenerency of matrix ω one can construct a nondegenerate two-form

$$\omega = \frac{1}{2}\omega^{ij}dx^i \wedge dx^j. \tag{1.28}$$

Jacobi identity implies closeness of this two-form

$$d\omega = 0. \tag{1.29}$$

A manifold equipped with such a form is called symplectic manifold and denoted by (M,ω). Obviously, M is an even-dimensional manifold. The Hamiltonian system is defined by a triple $(M,\omega,H(x))$, where $H(x)$ is a scalar function called Hamiltonian.

The Hamiltonian equations of motion generate a vector field that preserves the symplectic form ω

$$\frac{dx^i}{dt} = \{H, x^i\} = V^i_H : \qquad \mathcal{L}_{\mathbf{V}_H}\omega = 0, \tag{1.30}$$

where $\mathcal{L}_{\mathbf{V}}$ denotes Lie derivative along the vector field $\mathbf{V}$. Conversely, any vector field, preserving the symplectic structure, is locally a Hamiltonian vector field.

The vector field $\mathbf{V}$ defines a symmetry of Hamiltonian system,if it preserves both the Hamiltonian H and the symplectic form ω:

$$\mathcal{L}_{\mathbf{V}_{\mathcal{J}}}\omega = 0, \qquad \mathbf{V}_{\mathcal{J}} H = 0 \tag{1.31}$$

Hence,

$$\mathbf{V}_{\mathcal{J}} = \{\mathcal{J}, \quad \}, \qquad \{\mathcal{J}, H\} = 0. \tag{1.32}$$

The scalar function $\mathcal{J}(x)$ is called a constant of motion of Hamiltonian system.

1.3 INTEGRABLE SYSTEMS

Integrable models play a pivotal role in modern theoretical and mathematical physics. Their significance lies in the fact that distinct physical phenomena can often be described by similar mathematical frameworks. Consequently, exactly solvable models find applications across a wide range of domains. Through these models, a vast array of both macroscopic and microscopic physical phenomena can be effectively characterized. Furthermore, integrable models have implications beyond physics, as systems of integrable differential equations also arise in various other disciplines, including mathematics and computer science.

An N-dimensional mechanical system, characterized by N degrees of freedom, is called *integrable* if it possesses N constants of motion that mutually commute and are functionally independent [8, 9]. Moreover, the system may feature additional constants of motion. In such cases, we classify the system as *superintegrable.*Specifically, if an N-dimensional mechanical system exhibits $2N-1$ functionally independent constants of motion, it is called *maximally superintegrable* [1]. Conversely, if the system upholds $N+1$ conserved quantities, it is labelled *minimally superintegrable.* While integrability of models lets us easily separate variables just in a single coordinate system, with superintegrability, we can do this in many different coordinate systems. For instance, the two-dimensional oscillator is superintegrable, enabling us to separate variables using both Cartesian and polar coordinates. In classical mechanics, maximal superintegrability ensures trajectories remain closed. In the quantum mechanics, the energy

[1]According to Liouville's theorem in classical mechanics, an isolated mechanical system with N degrees of freedom can possess $2N-1$ additional constants of motion, aside from the Hamiltonian, that are in involution (meaning they mutually commute). This result is a consequence of the conservation of phase space volume in Hamiltonian dynamics.

spectrum of integrable models depends on N quantum numbers. For superintegrable systems possessing an additional K conserved quantities, the energy spectrum depends on $N-K$ quantum numbers. In cases of maximal superintegrability, the energy spectrum is determined by a single quantum number, leading to a degenerate energy spectrum at the quantum level.

Very well known examples of maximally superintegrable models are the N-dimensional Coulomb system and the N-dimensional harmonic oscillator, which are widely considered as an examples throughout this work. Both in chapter 3 and 4, as in bosonic case so in supersymmetrisation, this geometrical approach is demonstrated in these models. So it worth to give some basics on these systems, what we are going to do in the next two subsections.

1.3.1 OSCILLATOR

Harmonic oscillator is well known and maybe the most important example of a maximally superintegrable model [22]. Due to its simplicity and unique properties it plays a crucial role in all areas of modern physics. Techniques developed for harmonic oscillator can be used in all areas of physics, e. g. in condensed matter physics and quantum field theory. There are several extensions and generalizations of harmonic oscillator, namely non -harmonic oscillator, oscillator with additional potential. In current work oscillator is the key system. We will consider superintegrable generalizations of oscillator in curved spaces, for instance on spherical and pseudospherical spaces, Euclidean and projective complex manifolds. Extensions with additional potential will also be discussed, namely we will focus on superintegrable generalizations with an inverse square potential. Before discussing this generalizations it is important to discuss the standard harmonic oscillator.

The harmonic oscillator stands out as a well-known and arguably the most significant example of a maximally superintegrable model [22]. Due to simplicity and distinctive properties it plays a pivotal role across various domains in modern physics. Techniques that are developed for the harmonic oscillator find applications in diverse fields, including condensed matter physics and quantum field theory.Extensions and generalizations of the harmonic oscillator, such as the non-harmonic oscillator and oscillators with additional potential, exist. In our current study, the oscillator takes center stage as the key system. We explore the oscillator as a system that possesses Kähler phase space. Additionally, In the third chapter we discuss supersymmetrisation of the harmonic oscillator, now having, so to speak, Kähler phase superspace.Before delving into these generalizations, it is essential to lay the groundwork by discussing the standard harmonic oscillator.

N-dimensional harmonic oscillator is a system with quadratic potential and standard Poisson brackets.

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{\omega^2 x_i^2}{2}, \qquad \{p_i, x_j\} = \delta_{ij}, \quad \{p_i, p_j\} = \{x_i, x_j\} = 0 \tag{1.33}$$

Given the rotational symmetry of the system, angular momentum is conserved. It is well known that the symmetry associated with these conserved quantities is $SO(N)$.

$$L_{ij} = p_i x_j - p_j x_i, \qquad \{L_{ij}, L_{kl}\} = \delta_{il} L_{kj} - \delta_{kj} L_{il} + \delta_{jl} L_{ik} - \delta_{ik} L_{jl} \tag{1.34}$$

Moreover, oscillator has an additional conserved quantities that are quadratic on momenta

$$I_{ij} = p_i p_j + \omega^2 x_i x_j \tag{1.35}$$

This is the Fradkin tensor, and when combined with angular momentum, the system of conserved quantities for the harmonic oscillator possesses $U(N)$ symmetry. It is crucial to note that functional relations exist among these conserved quantities, leading to a total of $2N - 1$ functionally independent conserved quantities. The $U(N)$ symmetry becomes more apparent when we introduce complex quantities, which can be regarded as the classical analog of creation and annihilation operators.

$$u = \frac{p_i + i x_i}{\sqrt{2}}, \quad \bar{u} = \frac{p_i - i x_i}{\sqrt{2}}, \quad \{\bar{u}_i, u_j\} = i\delta_{ij} \tag{1.36}$$

In these coordinates, the Hamiltonian will exhibit a clear $U(N)$ invariance, allowing us to express conserved quantities as generators of this symmetry.

$$H = \sum_{i=1}^{N} u_i \bar{u}_i, \qquad M_{ij} = u_i \bar{u}_j \tag{1.37}$$

The energy spectrum can be expressed as follows, and as previously noted, it depends solely on one quantum number (n). [23].

$$E = \hbar\omega(n + \frac{N}{2}) \tag{1.38}$$

1.3.2 COULOMB PROBLEM

The Coulomb problem stands as another well-known example of a superintegrable model, significant in celestial mechanics and recognized for centuries. The system's symmetries, including angular momentum conservation (Kepler's second law) and the conservation of the Laplace-Runge-Lenz (or simply Runge-Lenz) vector, have been known for an extended period.

Within this thesis, we once again explore this superintegrable model in terms our geometrical language. The supersymmetrisation possibilities in this context are considered as well. However, the investigation of complex generalizations of the Coulomb system poses significant challenges and is not within the scope of our discussion. This is because the Coulomb problem exhibits orthogonal symmetries, while a complex structure necessitates unitary symmetry.

The Hamiltonian of N-dimensional Coulomb problem is

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} - \frac{\gamma}{r}, \qquad r = \sqrt{\sum_i x_i^2} \tag{1.39}$$

Poisson brackets are the same as given in(1.33). Again we have $SO(N)$ rotational symmetry and due to that angular momentum is a conserved quantity.

$$L_{ij} = p_i x_j - p_j x_i, \qquad \{L_{ij}, L_{kl}\} = \delta_{il} L_{kj} - \delta_{kj} L_{il} + \delta_{jl} L_{ik} - \delta_{ik} L_{jl} \tag{1.40}$$

We have additional constants of motion, which are called Runge-Lenz vector

$$A_i = L_{ij} p_j + \frac{\gamma x_i}{r}. \tag{1.41}$$

Together with angular momentum the system of conserved quantities has $SO(N+1)$ symmetry [24]. N-dimensional Coulomb problem can be obtained via reduction from free particle moving on $N+1$ dimensional sphere Since the symmetry of this system is obviously $SO(N+1)$, the symmetry of N-dimensional Coulomb problem is not surprising.

Again the number of independent constants of motion is $2N-1$. So the N-dimensional Coulomb system is maximally superintegrable.

The energy spectrum depends on one quantum number

$$E = -\frac{\gamma}{2\hbar^2 (n + \frac{N-3}{2})^2}. \tag{1.42}$$

1.4 SUPERSYMMETRIC MECHANICS

In this section we consider the Hamiltonian approach applied to classical supersymmetric mechanics. While supersymmetry was originally introduced in the quantum field theory, subsequent developments have revealed the intrinsic significance of supersymmetric mechanical models in modern physics. Firstly, considering mechanics as a one-dimensional field theory allows us to regard these models as a simple "toy" models of supersymmetric field theories and superstring theory. However, it is acknowledged that, to date, no empirical evidence substantiates the existence of supersymmetry in high-energy physics. In contrast, supersymmetry manifests in numerous physical quantum mechanical phenomena. For example, the widely recognized Landau problem can be interpreted as a paradigmatic supersymmetric model [20].

The third chapter of this thesis is devoted to the supersymmetric extensions of certain integrable models on Kähler manifolds. Therefore, it is useful to provide some fundamental information about supersymmetric mechanics. It is urgent to emphasize that Kähler structures play a pivotal role in supersymmetric field theoretical models. Notably, supersymmetric Lagrangians can be formulated by employing the Kähler potential to compose chiral superfields. [21].

Initially, we need to extend the concept of Poisson brackets to include odd Grassmann quantities. This extended structure is termed a "supersymplectic structure." It's essential to note that the Poisson brackets for two odd Grassmann quantities exhibit symmetry and resemble the anticommutator for operators in quantum mechanics, as discussed in [10]. Additionally, the Jacobi identity must be extended to accommodate this context.

$$\{f^{(a)}, g^{(b)}\} = -(-1)^{ab}\{g^{(b)}, f^{(a)}\} \tag{1.43}$$

$$(-1)^{ac}\{f^{(a)},\{g^{(b)},h^{(c)}\}\}+(-1)^{ab}\{g^{(b)},\{h^{(c)},f^{(a)}\}\}+(-1)^{bc}\{h^{(c)},\{f^{(a)},g^{(b)}\}\}=0 \tag{1.44}$$

where a, b, c take values 0 for even Grassmann variables and 1 for odd Grassmann variables.

We state that we have $\mathcal{N}=n$ supersymmetric mechanics when there exist n odd-Grassmann variables Q_i (supercharges) that satisfy the following relation

$$\{Q_i, Q_i\}=\delta_{ij}H, \qquad \{Q_i, H\}=0 \tag{1.45}$$

Given the one-dimensional field theory context, our superspace consists of time and Grassmann variables (t, θ_i), often referred to as "supertime." It is evident that this supersymmetry corresponds to the $\mathcal{N}=n, d=1$ SuperPoincaré algebra.

Let us examine the most straightforward example, namely $\mathcal{N}=1$ supersymmetric mechanics. In this scenario, any odd Grassmann variable can be selected, and its square can be identified as the Hamiltonian. However, due to its trivial nature, this case lacks significant interest.

Moving on to the $\mathcal{N}=2$ supersymmetric mechanics example. In this case, the supercharges can be redefined as $Q^{\pm}=(Q_1 \pm iQ_2)/\sqrt{2}$, resulting in the supersymmetric algebra taking the following form:

$$\{Q^+, Q^-\}=H, \qquad \{Q^+, Q^+\}=\{Q^-, Q^-\}=0 \tag{1.46}$$

If we consider a particle on a Riemannian manifold, the supercharges and the symplectic structure can be selected in the following form:

$$Q^{\pm}=(p_a \pm iW_{,a})\eta^a_{\pm}, \qquad \omega=dp_a\wedge dx^a+\frac{1}{2}R_{abcd}\eta^a_+\eta^b_- dx^c\wedge dx^d+g_{ab}D\eta^a_+\wedge D\eta^b_- \tag{1.47}$$

where $D\eta^a_{\pm}=d\eta^a_{\pm}+\Gamma^a_{bc}\eta^b_{\pm}dx^c$ and W is called superpotential. One can compute the Hamiltonian

$$H=\frac{1}{2}g^{ab}(p_a p_b+W_{,a}W_{,b})+W_{a;b}\eta^a_+\eta^b_-+R_{abcd}\eta^a_-\eta^b_+\eta^c_-\eta^d_+ \tag{1.48}$$

Chapter 2

Euler top and freedom in supersymmetrization of one-dimensional mechanics

2.1 INTRODUCTION

In this chapter, we explore a basic one-dimensional model that possesses a Kähler phase space. The model under consideration is an Euler top, a classical mechanical integrable system widely studied for its interesting dynamics. The Euler top represents a rotating rigid body, exhibiting rich behaviors related to angular momentum and precession, see, e.g. [1]. Conventionally it is described by the Hamiltonian system with degenerated Poisson brackets parameterized by the components of angular momentum $\ell = (x_1, x_2, x_3)$,

$$\{x_i, x_j\} = \varepsilon_{ijk} x_k, \qquad H = \sum_{i=1}^{3} \frac{x_i^2}{2I_i}, \tag{2.1}$$

where H is the Hamiltonian, and $I_i > 0$ are the principal momenta of inertia. Since x_i form $so(3)$ algebra, the system has a Casimir function

$$C = \sum_{i=1}^{3} x_i^2 \; : \quad \{C(x), x_i\} = 0 \tag{2.2}$$

Its fixation leads to the Hamiltonian system with two-dimensional **non-degenerated** phase space, i.e. one-dimensional system. Hence, Euler top is *a priori* integrable.

Being introduced centuries ago, Euler top has been studied as completely as the one-dimensional oscillator both at classical and quantum-mechanical levels. So, none of the open questions is to be studied there, except various aspects of its perturbations and generalizations.

Recently A.Galajinsky noticed the absence of the relevant supersymmetric extensions of Euler top [63]. He suggested its $\mathcal{N} = 1$ supersymmetrization via extension of the degenerated Poisson brackets (2.1) by *three real* Grassmann coordinates, stating that in general the resulting system lacks integrability. It seems to us that many questions asked in that paper come from the improper supersymmetrization procedure formulated in terms of degenerated phase space. As a consequence, it yields overcompleted number of fermionic variables which do not have impact on the actual properties of the system. Moreover, the invention of three Grassmann variables could yield the problems with the physical interpretation of the quantized version of system (though quantum aspects were not touched in that paper). Furthermore, the $\mathcal{N} \geq 2$ supersymmetric extensions of the Euler top, which at the quantum level create qualitative corrections to the initial spectrum (e.g. degeneracy of the energy levels etc), were not considered there at all.

In this chapter we propose to supersymmetrize the Euler top formulated in terms of nondegenerated phase space. First, fixing the value of the Casimir (2.2), we formulate the Hamiltonian system (2.1) in terms of two-dimensional nondegenerated phase space given by the complex projective plane $\mathbb{CP}^1$, i.e. as a one-dimensional mechanical system. Since any two-dimensional manifold can be provided with the Kähler structure, the initial system can be quantized by the well-developed technique of geometric quantization on Kähler manifold (see, e.g. [64]).

Then we present the procedure of $\mathcal{N} = 2k$ supersymmerization of the systems with generic two-dimensional nondegenerated phase space which results in *á priory* integrable supersymmetric extension of the initial system. Suggested procedure provides us with the family of the supersymmetric systems parameterized by the $\mathcal{N}/2$ angle-like arbitrary functions. Similar functional freedom was noticed earlier in the one-dimensional $\mathcal{N}=2$ supersymmetric mechanics [42] and in the two-dimensional $\mathcal{N}=4$ supersymmetric ones [65]. In proposed supersymmetrization scheme the fermionic variables are splitted from the bosonic ones, in contrast with $\mathcal{N}=2$ supersymmetrization procedure of the systems with generic Kähler phase space suggested in [66]. They form, with respect to the Poisson brackets, the Clifford algebra. Thus, guantization of the supersymmetric system is straightforward: we should perform geometric quantization of the initial bosonic system and then replace the fermionic variables by the respective gamma-matrices.

The chapter is organized as follows.

In *Section 2.2* we review the description of Euler top on the phase space given by the complex projective plane $\mathbb{CP}^1$.

In *Section 2.3* we present the $\mathcal{N}=2k$ supersymmetization procedure for the system with generic one-(complex)dimensional Kähler spaces.

In *Section 2.4* we summarize obtained results and discuss the possible extensions of the proposed scheme to the Lagrange and Kowalewski tops.

2.2 EULER TOP

For the description of the Euler top (2.1) in terms of the non-degenerated phase space, let us introduce instead of x_i, the coordinates $j, z, \bar{z}$

$$j := \sqrt{\sum_{i=1}^{3} x_i^2}, \qquad z := \frac{x_1 + \imath x_2}{j - x_3}. \tag{2.3}$$

Clearly, j is the complete angular momentum.

In these coordinates the Poisson brackets read

$$\{\bar z, z\} = -\frac{\imath}{2j}(1 + z\bar z)^2, \quad \{z, j\} = \{z, z\} = 0, \tag{2.4}$$

while the momentum generators look as follows

$$j^2 := \sum_{i=1}^{3} x_i^2, \tag{2.5}$$

$$x_1 := h_1 = j\frac{z + \bar z}{1 + z\bar z}, \quad x_2 := h_2 = j\frac{\imath(\bar z - z)}{1 + z\bar z}, \quad x_3 := h_3 = j\frac{z\bar z - 1}{1 + z\bar z}. \tag{2.6}$$

However, the point $x_3 = j$ cannot be described in these terms.To improve this lack we should introduce, instead of z, another coordinates $\tilde z, \bar{\tilde z}$

$$\tilde z := \frac{x_1 - \imath x_2}{j + x_3} \ : \quad \{\bar{\tilde z}, \tilde z\} = -\frac{\imath}{2j}(1 + \tilde z\bar{\tilde z})^2. \tag{2.7}$$

Out of the points $x_3 = \pm j$ these coordinates are related with each other as follows

$$\tilde z = \frac{1}{z}. \tag{2.8}$$

The Poisson brackets (2.4) and (2.7) transform to each other under this transformation. Thus, fixing j to be constant we arrive at the two-dimensional phase space covered by two charts (parameterized by the single complex coordinate z or $\tilde z$) and equipped with the one-(complex)dimensional Kähler structure - the complex projective plane $\mathbb{CP}^1$ with the Fubini-Study metrics

$$g(z, \bar z)dzd\bar z := 2j\frac{dzd\bar z}{(1 + z\bar z)^2}, \tag{2.9}$$

which corresponds to the Kähler potential

$$K(z, \bar z) = 2j\log(1 + z\bar z). \tag{2.10}$$

The generators (2.6) become Killing potentials of $\mathbb{CP}^1$, and define the Hamiltonian holomorphic vector fields,

$$
\begin{aligned}
\{h_1, \quad \} &= \imath(1-z^2)\partial_z + c.c.,\\
\{h_2, \quad \} &= -(1+z^2)\partial_z + c.c.,\\
\{h_3, \quad \} &= 2\imath z\partial_z + c.c.,
\end{aligned}
\tag{2.11}
$$

In these terms the Hamiltonian of Euler top reads

$$
H = \sum_{i=1}^{3} \frac{x_i^2}{2I_i} = -j^2 \frac{b(z^2+\bar{z}^2) + 2az\bar{z}}{2(1+z\bar{z})^2} + \frac{j^2}{2I_3}, \tag{2.12}
$$

where

$$
a := \frac{2}{I_3} - \frac{1}{I_1} - \frac{1}{I_2}, \qquad b := \frac{1}{I_2} - \frac{1}{I_1}. \tag{2.13}
$$

Now, let us rewrite the Euler top in canonical coordinates. For this purpose we notice that $\mathbb{CP}^1$ is just the two-dimensional sphere $\mathbf{S}^2$ formulated in the projective coordinates

$$
z = \cot\frac{\theta}{2} \mathrm{e}^{\imath\varphi}, \tag{2.14}
$$

where θ, φ are the spherical coordinates. In these terms the Poisson bracket (2.4) reads

$$
\{\varphi, j\cos\theta\} = 1 \tag{2.15}
$$

Hence, the function $p := j\cos\theta$ defines the canonical momentum conjugated to φ. In terms of these canonical coordinates the angular momentum generators (and Killing potentials) (2.6) read

$$
x_1 + \imath x_2 = j\sin\theta \mathrm{e}^{\imath\varphi} = \sqrt{j^2 - p^2}\mathrm{e}^{\imath\varphi}, \quad x_3 = j\cos\theta = p, \tag{2.16}
$$

while the Hamiltonian of the Euler top takes the form

$$
H = \frac{1}{4}(a + b\cos 2\varphi)p^2 + \frac{j^2}{4}\left(\frac{2}{I_3} - (a + b\cos 2\varphi)\right). \tag{2.17}
$$

Without loss of generality we assume

$$
I_3 \le I_2 \le I_1, \tag{2.18}
$$

and perform canonical transformation $(p,\varphi)\to(P,Q)$, where

$$, \tag{2.19}$$

$$\begin{aligned} P &= \sqrt{\frac{a+b\cos 2\varphi}{2}}\,p, \\ Q &= \sqrt{\frac{2}{a+b}}\int\frac{d\varphi}{\sqrt{1-\frac{2b}{a+b}\sin^2\varphi}} = \sqrt{\frac{2}{a+b}}F\left(\varphi,k\right): \quad \{Q,P\}=1, \end{aligned} \tag{2.20}$$

where $F\left(\varphi,k\right)$ is an elliptic integral of the first kind, with $k=\sqrt{2b/(a+b)}$ being its modulus, and φ is the so-called Jacobi amplitude (which defines the Jacobi elliptic functions) [67],

$$\varphi = F^{-1}\left(F,k\right) = \mathbf{amp}\left(F,k\right). \tag{2.21}$$

Respectively

$$\sin\varphi = \sin(\mathbf{amp}(F,k)) = \mathbf{sn}(F,k) \tag{2.22}$$

is the Jacobi sine amplitude of the elliptic functions.

In this terms the Hamiltonian of Euler top reads

$$H = \frac{1}{2}P^2 + \frac{j^2 b}{2}\mathbf{sn}^2\left(\sqrt{\frac{a+b}{2}}Q, \sqrt{\frac{2b}{a+b}}\right) + \frac{j^2}{2I_1}. \tag{2.23}$$

In the particular case of the symmetric top ($I_1 = I_2 := I$) it reduces to the one-dimensional free particle Hamiltonian

$$H = \frac{1}{2}\left(\frac{1}{I_3}-\frac{1}{I}\right)p^2 + \frac{j^2}{2I}. \tag{2.24}$$

So, the Euler top is the one-dimensional Hamiltonian system with $\mathbb{CP}^1$ phase space and with the Hamiltonian given by the quadratic functions of its Killing potentials. In the canonical coordinates it results in the one-dimensional nonlinear oscillator.

2.3 SUPERSYMMETRY

In the previous section we formulated the Euler top in terms of one-(complex)dimensional phase space given by complex projective plane. Being one-dimensional system, the Euler top allows many ways of supersymmetrization, including supersymmetrization in canonical coordinates. However, we are interested in the supersymmetrization compatible with the Kähler geometry describing the phase space of the Euler top.

One of the ways to supersymmetrize the Euler top is to use the approach suggested in [66] which is based on the extension of the Kähler phase space to the super-Kähler one defined by the potential

$$\mathcal{K}(z, \bar{z}, \theta_a, \bar{\theta}^a) = K(z, \bar{z}) + F(\imath g(z, \bar{z})\theta_a\bar{\theta}^a), \tag{2.25}$$

where $F(x)$ is the real function with $F'(0) \neq 0$, with $K(z, \bar{z})$, $g(z, \bar{z})$ given by (2.10) and (2.9), while the fermionic variables θ_a are associated with dz, in complete similarity with the superfield approach.

Another particular way of supersymmetrization is to extend the complex projective plane to the complex projective super-plane given by the Kähler potential

$$\tilde{K}(z, \bar{z}, \theta_a, \bar{\theta}^a) = 2j \log(1 + z\bar{z} + \theta_a\bar{\theta}^a). \tag{2.26}$$

Such approach has been taken in [9], where it was applied to the Lobachevsky plane (i.e. non-compact version of complex projective plane) for the construction of $\mathcal{N}$-extended one-dimensional superconformal mechanics. Later on, this approach was generalized to the higher-dimensional systems in [19].

Below we suggest different, less geometric approach, which is applicable not only for the Euler top, but for any one-dimensional system. We will consider the systems with generic two-(real)dimensional phase space. Such phase spaces can be always equipped with the one-(complex)dimensional Kähel structure, so that the Poisson brackets will be given by the relation

$$\{z, \bar{z}\} = \frac{\imath}{g(z, \bar{z})}. \tag{2.27}$$

For the construction of $\mathcal{N}$-supersymmetric extensions of these systems (with even $\mathcal{N}$) we extend this phase space by the canonical complex Grassmann variables ψ_a, $a = 1, \ldots, \frac{\mathcal{N}}{2}$

$$\left\{\psi_a, \bar{\psi}^b\right\} = \imath\delta_a^b, \tag{2.28}$$

where $\overline{(\psi_a)} := \bar{\psi}^a$ and $\overline{\mathcal{F}_1\mathcal{F}_2} = \overline{\mathcal{F}}_2\overline{\mathcal{F}}_1$.

With these Poisson brackets at hands we can construct the $\mathcal{N}$ supersymmetric extensions of two-dimensional systems defined by the Poisson brackets (2.27) and by any positive Hamiltonian $H(z, \bar{z}) > 0$,

$$\left\{Q_a, \overline{Q}^b\right\} = \imath\delta_a^b\mathcal{H}\ , \qquad \mathcal{H} := H(z, \bar{z}) + \text{ fermions}. \tag{2.29}$$

In accordance with the generalization of Liouville theorem to the supermanifolds [68](see also [42]) these supersymmetric extensions will be *apriory* integrable, since we will get the system with $(2|\mathcal{N})_{\mathbf{R}}$-dimensional phase space with one bosonic constant of motion $\mathcal{H}$ and $\mathcal{N}$ fermionic constants of motion $Q_a, \overline{Q}^b$ commuting with the bosonic integral $\mathcal{H}$.

$\mathcal{N} = 2$ SUPERSYMMETRY

For the construction of $\mathcal{N} = 2$ supersymmetric extension of the system with Hamiltonian $H(z, \bar{z}) > 0$ we choose, following [42], the appropriate Ansatz for supercharges and arrive the family of $\mathcal{N} = 2$ supersymmetric extensions of the Hamiltonian H, parameterized by the arbitrary real function $\Phi(z, \bar{z})$

$$Q = \sqrt{H}\mathrm{e}^{\imath\Phi}\psi, \qquad \overline{Q} = \sqrt{H}\mathrm{e}^{-\imath\Phi}\bar{\psi} \quad \Rightarrow \quad \mathcal{H} = H + \{\Phi, H\}\psi\bar{\psi}. \tag{2.30}$$

Specifying the Poisson brackets and Hamiltonian we will get the respective supersymmetric extension of the Euler top.

Direct extension of construction to the $\mathcal{N} \geq 4$ supersymmetric mechanics fixes the function Φ and leads to the trivial family of the supersymmetric Hamiltonians. Namely, choosing $Q_a = \sqrt{H}\mathrm{e}^{\imath\Phi}\psi_a$, we get that the superalgebra (2.29) is fulfilled when $\{H, \Phi\} = 0$. Hence, the resulting supersymmetric Hamiltonian is trivial: it coincides with the initial bosonic Hamiltonian.

$\mathcal{N} = 4$ SUPERSYMMETRY

For the construction of nontrivial $\mathcal{N} = 4$ supersymmetric system we choose the following Ansatz for supercharges

$$
\begin{aligned}
Q_a &= f_1(z, \bar{z})\psi_a + f_2(z, \bar{z})\psi_a \sum_{b=1}^{\mathcal{N}/2} \psi_b \bar{\psi}^b, \\
\overline{Q}^a &= \bar{f}_1(z, \bar{z})\bar{\psi}^a - \bar{f}_2(z, \bar{z})\bar{\psi}^a \sum_{b=1}^{\mathcal{N}/2} \bar{\psi}^b \psi_b \ ,
\end{aligned}
\tag{2.31}
$$

with

$$f_1(z, \bar{z}) := \sqrt{H}\mathrm{e}^{\imath\Phi_1(z,\bar{z})}, \tag{2.32}$$

$$f_2 = R(z, \bar{z})\mathrm{e}^{\imath(\Phi_1 - \Phi_2)}, \tag{2.33}$$

$$\bar{R} = R, \tag{2.34}$$

$$\bar{\Phi}_a(z, \bar{z}) = \Phi_a(z, \bar{z}). \tag{2.35}$$

Then, we require that the supercharges (2.3) form the $\mathcal{N} = 4$ Poincaré superalgebra (2.29), which results in the following conditions on the functions involved

$$\imath\left\{f_1,\bar{f}_1\right\} = f_1\bar{f}_2 + \bar{f}_1 f_2 \quad \Leftrightarrow \quad \{\sqrt{H},\Phi_1\} = R\cos\Phi_2, \tag{2.36}$$

with the Hamiltonian $\mathcal{H}$ acquiring the form

$$\begin{aligned}
\mathcal{H} &= f_1\bar{f}_1 + \imath\left\{f_1,\bar{f}_1\right\}\sum_{a=1}^{\mathcal{N}/2}\psi_a\bar{\psi}^a + \frac{\imath}{2}\left(\left\{f_1,\bar{f}_2\right\} + \left\{f_2,\bar{f}_1\right\}\right)\left(\sum_{a=1}^{\mathcal{N}/2}\psi_a\bar{\psi}^a\right)^2 \\
&= H + \{H,\Phi_1\}\psi_a\bar{\psi}^a + A(\sqrt{H},\Phi_{1,2})\left(\sum_{a=1}^{\mathcal{N}/2}\psi_a\bar{\psi}^a\right)^2,
\end{aligned} \tag{2.37}$$

with

$$\begin{aligned}
A(\sqrt{H},\Phi_{1,2}) := & \frac{\imath}{2}\left(\left\{f_1,\bar{f}_2\right\} + \left\{f_2,\bar{f}_1\right\}\right) \\
= & \left(\{\sqrt{H},\Phi_1\}\right)^2 - \frac{\{\sqrt{H},\Phi_1\}\{\sqrt{H},\Phi_2\}}{\cos^2\Phi_2} \\
& + \left\{\{\sqrt{H},\Phi_1\},\Phi_1\right\}\sqrt{H} + \left\{\{\sqrt{H},\Phi_1\},\sqrt{H}\right\}\tan\Phi_2.
\end{aligned} \tag{2.38}$$

Thus, we get the $\mathcal{N}=4$ supersymmetric mechanics parametrized by two arbitrary functions $\Phi_{1,2}$.

We can use the Ansatz (2.3) for the construction of $\mathcal{N}>4$ supersymmetric systems as well. However, in that case we get the additional constraints on the functions f_1, f_2,

$$\mathcal{N}=6 \quad : \quad \{f_1,\bar{f}_2\} + \{f_2,\bar{f}_1\} = 2\imath f_2\bar{f}_2 \tag{2.39}$$

$$\mathcal{N}=8 \quad : \quad \{f_1,\bar{f}_2\} + \{f_2,\bar{f}_1\} = 2\imath f_2\bar{f}_2, \qquad \{f_2,\bar{f}_2\} = 0. \tag{2.40}$$

These constraints, obviously, restrict the functional freedom existing in $\mathcal{N}=4$ systems.

In the $\mathcal{N}=6$ case this constraint leads to the following restriction

$$A(\sqrt{H},\Phi_a) + \left(\frac{\{\sqrt{H},\Phi_1\}}{\cos\Phi_2}\right)^2 = 0, \tag{2.41}$$

with A given by (2.38). Hence, the system has single functional degree of freedom parameterized by Φ_1, as in the $\mathcal{N}=2$ case.

The requirement of $\mathcal{N}=8$ supersymmetry further fixes the value of Φ_1

$$\{\{\sqrt{H},\Phi_1\},\Phi_1-\Phi_2\}=\{\Phi_1,\Phi_2\}\{\sqrt{H},\Phi_1\}\tan\Phi_2. \tag{2.42}$$

As a result, we get the $\mathcal{N}=8$ Hamiltonian with no functional freedom.

The evident way to construct the $\mathcal{N}>4$ systems with wide functional freedom is to extend the supercharges Ansatz by higher 5- and 7- fermionic terms .

2.4 $\mathcal{N}=6,8$ SUPERSYMMETRIC MECHANICS

In the previous section we have shown that the supercharges with cubic fermionic terms allow to construct $\mathcal{N}=4$ supersymmetric mechanics with two functional degrees of freedom, $\mathcal{N}=6$ supersymmetric mechanics with single functional degree of freedom, and $\mathcal{N}=8$ supersymmetric mechanics without any functional freedom.

One can guess that the supercharges with fifth-order fermionic term could lead to the $\mathcal{N}=6$ supersymmetric mechanics with three functional degrees of freedom and to $\mathcal{N}=8$ supersymmetric mechanics with two functional degrees of freedom. Furthermore, one can expect that the supercharges with seventh-order fermionic terms could lead to the $\mathcal{N}=8$ supersymmetric mechanics with four functional degrees of freedom and so on. Let us show that it is indeed the case.

In order to construct the $\mathcal{N}=6$ supersymmetric systems with three functional degrees of freedom we consider the following Ansatz for the supercharges

$$\begin{aligned}
Q_a &= f_1(z,\bar z)\psi_a + f_2(z,\bar z)\psi_a\left(\sum_{b=1}^{\mathcal{N}/2}\psi_b\bar\psi^b\right) + f_3(z,\bar z)\psi_a\left(\sum_{b=1}^{\mathcal{N}/2}\psi_b\bar\psi^b\right)^2,\\
\overline{Q}^a &= \bar f_1(z,\bar z)\bar\psi^a + \bar f_2(z,\bar z)\bar\psi^a\left(\sum_{b=1}^{\mathcal{N}/2}\psi_b\bar\psi^b\right) + \bar f_3(z,\bar z)\bar\psi_a\left(\sum_{b=1}^{\mathcal{N}/2}\psi_b\bar\psi^b\right)^2,
\end{aligned} \tag{2.43}$$

with $a,b=1,2,3$.

Then, requiring that these functions form $\mathcal{N}=6$ Poincaré superalgebra (2.29), we get the following restrictions to the functions f_a

$$
\begin{aligned}
& f_1\bar{f}_2+\bar{f}_1 f_2-\imath\left\{f_1,\bar{f}_1\right\}=0, \\
& 2f_3\bar{f}_1+2f_2\bar{f}_2+2f_1\bar{f}_3-\imath\left\{f_1,\bar{f}_2\right\}+\imath\left\{\bar{f}_1,f_2\right\}=0,
\end{aligned}
\tag{2.44}
$$

The respective Hamiltonian then reads

$$
\begin{aligned}
\mathcal{H}= & \frac{1}{2}f_1\bar{f}_1+\frac{1}{2}(f_1\bar{f}_2+\bar{f}_1 f_2)\sum_{a=1}^{\mathcal{N}/2}\psi_a\bar{\psi}^a \\
& +\frac{1}{2}(f_3\bar{f}_1+f_1\bar{f}_3+f_2\bar{f}_2)\left(\sum_{a=1}^{\mathcal{N}/2}\psi_a\bar{\psi}^a\right)^2 \\
& +\frac{1}{2}(f_2\bar{f}_3+f_3\bar{f}_2)\left(\sum_{a=1}^{\mathcal{N}/2}\psi_a\bar{\psi}^a\right)^3
\end{aligned}
\tag{2.45}
$$

Representing f_a in the form

$$
f_1=\sqrt{H}\mathrm{e}^{\imath\Phi_1},\quad f_2=R_2\mathrm{e}^{\imath(\Phi_1-\Phi_2)},\quad f_3=R_3\mathrm{e}^{\imath(\Phi_1-\Phi_2-\Phi_3)},
\tag{2.46}
$$

and re-writing in these terms the conditions (2.4), we conclude that the functions Φ_1,Φ_2,Φ_3 remains unfixed. Therefore, we arrive at the family of $\mathcal{N}=6$ supersymmetric mechanics parameterized by three arbitrary real functions.

In order to construct the $\mathcal{N}=8$ supersymmetric systems with four functional degrees of freedom we introduce the following generalization of the Ansatz (2.3),

$$Q_a = f_1(z,\bar z)\psi_a + f_2(z,\bar z)\psi_a \sum_{b=1}^{\mathcal{N}/2}\psi_b\bar\psi^b + f_3(z,\bar z)\psi_a \left(\sum_{b=1}^{\mathcal{N}/2}\psi_b\bar\psi^b\right)^2 + f_4(z,\bar z)\psi_a \left(\sum_{b=1}^{\mathcal{N}/2}\psi_b\bar\psi^b\right)^3,$$

$$\overline{Q}^a = \bar f_1(z,\bar z)\bar\psi^a + \bar f_2(z,\bar z)\bar\psi^a \sum_{b=1}^{\mathcal{N}/2}\psi_b\bar\psi^b + \bar f_3(z,\bar z)\bar\psi_a \left(\sum_{b=1}^{\mathcal{N}/2}\psi_b\bar\psi^b\right)^2 + \bar f_4(z,\bar z)\bar\psi_a \left(\sum_{b=1}^{\mathcal{N}/2}\psi_b\bar\psi^b\right)^3, \quad (2.47)$$

with $a,b = 1,2,3,4$.

Then, requiring that these functions form $\mathcal{N}=8$ Poincaré superalgebra (2.29), we get the following restrictions on the functions f_a

$$f_1\bar f_2 + \bar f_1 f_2 - \imath\left\{f_1,\bar f_1\right\} = 0, \quad (2.48)$$

$$2f_3\bar f_1 + 2f_2\bar f_2 + 2f_1\bar f_3 - \imath\left\{f_1,\bar f_2\right\} + \imath\left\{\bar f_1, f_2\right\} = 0, \quad (2.49)$$

$$3f_4\bar f_1 + 3f_3\bar f_2 + 3f_2\bar f_3 + 3f_1\bar f_4 - \imath\left\{f_1,\bar f_3\right\} + \imath\left\{\bar f_1, f_3\right\} - \imath\left\{f_2,\bar f_2\right\} = 0. \quad (2.50)$$

The respective Hamiltonian then reads

$$\mathcal{H} = \frac{1}{2}f_1\bar f_1 + \frac{1}{2}(f_1\bar f_2 + \bar f_1 f_2)\left(\sum_{a=1}^{\mathcal{N}/2}\psi_a\bar\psi^a\right) + \frac{1}{2}(f_3\bar f_1 + f_1\bar f_3 + f_2\bar f_2)\left(\sum_{a=1}^{\mathcal{N}/2}\psi_a\bar\psi^a\right)^2 + \frac{1}{2}(f_4\bar f_1 + f_1\bar f_4 + f_2\bar f_3 + f_3\bar f_2)\left(\sum_{a=1}^{\mathcal{N}/2}\psi_a\bar\psi^a\right)^3 + \frac{\imath}{8}(\left\{f_1\,,\bar f_4\right\} + \left\{f_2\,,\bar f_3\right\} + \left\{f_3\,,\bar f_2\right\} + \left\{f_4\,,\bar f_1\right\})\left(\sum_{a=1}^{\mathcal{N}/2}\psi_a\bar\psi^a\right)^4 \quad (2.51)$$

Let us notice that the restriction (3.33) for $\mathcal{N}=8$ system coincides with the restrictions (2.4) for $\mathcal{N}=6$ case. While the additional constraint (??) contains extra complex function $f_4(z,\bar{z})$. Hence, representing f_a in the form

$$f_1 = \sqrt{H}\mathrm{e}^{\imath\Phi_1}, \tag{2.52}$$

$$f_2 = R_2\mathrm{e}^{\imath(\Phi_1-\Phi_2)}, \tag{2.53}$$

$$f_3 = R_3\mathrm{e}^{\imath(\Phi_1-\Phi_2-\Phi_3)}, \tag{2.54}$$

$$f_4 = R_4\mathrm{e}^{\imath(\Phi_1-\Phi_2-\Phi_3-\Phi_4)}, \tag{2.55}$$

we conclude that the functions $\Phi_1,\ldots,\Phi_4$ remain unfixed. Therefore, the $\mathcal{N}=8$ supersymmetric Hamiltonian (2.51) depends on four arbitrary real functions.

So, specifying the formulae given in the Third and Fourth Sections to the particular case of Euler top given in the Section 2 by (2.9) *we will get its integrable* $\mathcal{N}=2,4,6,8$ *supersymmetric extensions.*

From the consideration above it is easy to deduce that for the construction of $\mathcal{N}=10,12,\ldots 2k$ superextensions of initial Hamiltonian we should choose the following ansatzes for the supercharges

$$Q_a = f_1(z,\bar{z})\psi_a + \sum_{l=1}^{\mathcal{N}/2} f_{l+1}(z,\bar{z})\psi_a \left(\sum_{b=1}^{\mathcal{N}/2}\psi_b\bar{\psi}^b\right)^l,$$

$$\overline{Q}^a = \bar{f}_1(z,\bar{z})\bar{\psi}^a + \sum_{l=1}^{\mathcal{N}/2} \bar{f}_{l+1}(z,\bar{z})\bar{\psi}^a \left(\sum_{b=1}^{\mathcal{N}/2}\psi_b\bar{\psi}^b\right)^l, \tag{2.56}$$

with $a,b=1,\ldots,\mathcal{N}/2k$. Then, requiring that they form Poincaré superalgebra (2.29) we will get the family of $\mathcal{N}=2k$ supersymmetric Hamiltonians parameterized by k arbitrary real functions.

2.5 CONCLUSION

In this chapter, we have shown, firstly, that the Euler top can be represented as a one-dimensional system with a Kähler phase space $\mathbb{CP}^1$.

Then, we suggest a generic procedure for $\mathcal{N} = 2k$ supersymmetrization of a generic one-dimensional system. Hence, we have provided an entire family of supersymmetric extensions of the Euler top as well.

Further possible developments, namely potential applications to Kowalevski and Lagrange tops, are discussed in the last chapter.

Chapter 3

Non-compact Complex Projective Space as a Phase Space of Superintegrable Systems

3.1 INTRODUCTION

The so called "non-compact complex projective space", denoted here as $\widetilde{\mathbb{CP}}^N$, is a non-compact analogue of complex projective space. The reason to study particularly non-compact space is due yo the fact that it naturally leads us to formulation of standard conformal mechanics, and its deformations us an example of integrable and superintegrable systems. Bellow is adopted the Hamiltonian reduction approach while constructing projective spaces. Both, compact ($\mathbb{CP}^N$) and non-compact ($\widetilde{\mathbb{CP}}^N$) cases are considered in parallel, with the upper sign presenting the compact case, the lower one non-compact. N-dimensional complex projective space $\mathbb{CP}^N$, compact and non-compact, can be obtained by the reduction from Euclidean complex space $\mathbb{C}^{N+1}$ and pseudo-Euclidean complex space $\mathbb{C}^{1.N}$ respectively. The metric and Kähler potentials of $\mathbb{C}^{N+1}$ and $\mathbb{C}^{1.N}$ are given by

$$ds^2 = g_{i\bar{j}} du^i d\bar{u}^j \equiv du^0 d\bar{u}^0 \pm du^a d\bar{u}^a \equiv du^0 d\bar{u}^0 \pm du d\bar{u}, \tag{3.1}$$

where the indices like $i, j, k, ...$ are running from 0 to N and ones like $a, b, c, ...$ from 1 to N. $g_{i\bar{j}}$ is the Riemann metric, given by

$$g_{i\bar{j}} = \begin{cases} g_{i\bar{j}} = \delta_{i\bar{j}}, & \text{for compact case} \\ g_{i\bar{j}} = \gamma_{i\bar{j}}, & \text{for non-compact case} \end{cases}, \tag{3.2}$$

where $\delta_{i\bar{j}}$ is Kronecker symbol and $\gamma = \mathrm{diag}(1, -1, ..., -1)$. Kähler structures giving these metrics and associated Poisson brackets are

$$\mathcal{K}(u, \bar{u}) = u^0 \bar{u}^0 \pm u^a \bar{u}^a, \quad \omega = -\imath g_{i\bar{j}} du^i \wedge d\bar{u}^j, \quad \{u^i, \bar{u}^j\} = \imath g^{i\bar{j}}, \tag{3.3}$$

where $\mathcal{K}$ is the Kähler potential and ω is an antisymmetric 2-form. $g^{i\bar{j}}$ is the inverse metric in these trivial cases coincides with metric.

The isometry algebras of $\mathbb{C}^{N+1}$ and $\mathbb{C}^{1.N}$ are $u(N+1) = u(1) \times su(N+1)$ and $u(1.N) = u(1) \times su(1.N)$ respectively.The $u(1)$ generator(s), which form the center of the respective algebra, are given by $J = u^0 \bar{u}^0 \pm u^a \bar{u}^a$. To obtain the compact and non-compact complex projective spaces from the compact and non-compact Euclidean complex spaces by Hamiltonian reduction one needs to do following. To reduce the system by the action of generator J, first we have to choose $2N$ real (N complex) functions commuting with J. Then, we have to calculate their Poisson brackets and fix the $u(1)$ generator and hence the level surface by

$$J = g, \tag{3.4}$$

where g is chosen to be a positive constant for further convenience. The role of such functions, which constitute the coordinates of the newly reduced $\mathbb{CP}^N$ and $\widetilde{\mathbb{CP}}^N$, can be as follows (for both, compact and non-compact case):

$$z^a = \frac{u^a}{u^0}, \quad \left(\bar{z}^a = \frac{\bar{u}^a}{\bar{u}^0}\right). \tag{3.5}$$

So in this way we arrive at the most known examples of nontrivial Kähler manifolds, namely, the N-dimensional complex projective space $\mathbb{CP}^N$ and its non-compact analog $\widetilde{\mathbb{CP}}^N$. They can be equipped with the $su(N+1)$-invariant (for the compact case) and the $su(N.1)$ invariant (for the non-compact case) Kähler metrics, known as the Fubini-Study metrics. These metrics and respective Kähler potentials are defined by the expressions (again, with the upper sign corresponding to $\mathbb{CP}^N$, and the lower sign to $\widetilde{\mathbb{CP}}^N$) [1]

$$g_{a\bar{b}}dz^a d\bar{z}^b = \frac{gdzd\bar{z}}{1 \pm z\bar{z}} \mp \frac{g(\bar{z}dz)(zd\bar{z})}{(1 \pm z\bar{z})^2}, \quad \mathcal{K} = \pm g \log(1 \pm z\bar{z}), \tag{3.6}$$

as well as the inverse metrics and Poisson brackets given by them

$$g^{\bar{a}b} = (1 \pm z\bar{z})(\bar{z}^a z^b \pm \delta^{\bar{a}b}), \qquad \{z^a, \bar{z}^b\} = \frac{\imath}{g}(1 \pm z\bar{z})(z^a\bar{z}^b \pm \delta^{a\bar{b}}). \tag{3.7}$$

Let us notice that the complex projective space $\mathbb{CP}^N$ is defined by $N+1$ charts, while its noncompact analog $\widetilde{\mathbb{CP}}^N$ by a single chart. Moreover, in the latter case the range of validity of the coordinates z^a is

$$|z^a| < 1, \quad \sum_{a=1}^{N} z^a \bar{z}^a < 1 \tag{3.8}$$

The isometries of projective spaces are defined by the Killing potentials

$$h_{a\bar{b}} = g\frac{\bar{z}^a z^b \mp \delta^{\bar{a}b}}{1 \pm z\bar{z}}, \qquad h_a = g\frac{2\bar{z}^a}{1 \pm z\bar{z}}, \qquad h_{\bar{a}} = g\frac{2z^a}{1 \pm z\bar{z}}. \tag{3.9}$$

These generators form the $su(N+1)$ algebra for the upper sign, and the $su(N.1)$ for the lower sign (the generators $h_{a\bar{b}}$ form $u(N)$ algebra):

$$\begin{aligned}
&\{h_a, h_b\} = 0,\\
&\{h_a, h_{\bar{b}}\} = -4\imath h_{a\bar{b}},\\
&\{h_a, h_{b\bar{c}}\} = \pm\imath\left(\delta_{a\bar{c}}h_b + \delta_{b\bar{c}}h_a\right),\\
&\{h_{a\bar{b}}, h_{c\bar{d}}\} = \pm\imath(\delta_{a\bar{d}}h_{c\bar{b}} - \delta_{\bar{b}c}h_{a\bar{d}}).
\end{aligned} \tag{3.10}$$

[1] Through this text we use the notation $z\bar{z} \equiv \sum_{c=1}^{N} z^c\bar{z}^c$, $zd\bar{z} \equiv \sum_{c=1}^{N} z^c d\bar{z}^c$ etc.

3.2 POINCARÉ AND KLEIN MODELS OF THE LOBACHEVSKY PLANE

The one-dimensional noncompact complex projective space $\widetilde{\mathbb{CP}}^1$ is the Lobachevsky plane (upper sheet of two-sheet hyperboloid) proper. Its Fubini-Study metrics results in the $su(1.1) = so(1.2)$-invariant Kähler metric parameterized by the unit disc of two-dimensional plane, which is known as Poincaré model [57]

$$ds^2 = \frac{gdzd\bar{z}}{(1 - z\bar{z})^2}, \quad \Rightarrow \quad \mathcal{K} = -g\log(1 - z\bar{z}), \qquad |z| < 1. \tag{3.11}$$

In this particular case the Killing potentials read

$$h = g\frac{1 + z\bar{z}}{1 - z\bar{z}}, \quad h_+ = g\frac{2\bar{z}}{1 - z\bar{z}}\ , \quad h_- = g\frac{2z}{1 - z\bar{z}}\ : \tag{3.12}$$

$$\{h_+, h_-\} = -4\imath h, \quad \{h_\pm, h\} = \mp 2\imath h_\pm. \tag{3.13}$$

Performing the transformation

$$z = \frac{1 - \imath w}{1 + \imath w} \tag{3.14}$$

we arrive at the so-called Klein model parameterized by lower two-dimensional half-plane [57]

$$ds^2 = \frac{gdwd\bar{w}}{[\imath(w - \bar{w})]^2}, \quad \mathcal{K} = -g\log[\imath(w - \bar{w})], \qquad \mathrm{Im}\, w < 0. \tag{3.15}$$

We could have obtained this one dimensional manifold $\widetilde{\mathbb{CP}}^1$ directly by reduction of pseudo-Euclidean space (which the Lobachevsky plane in this case), if only we took the Klein model of it, namely taking the metric

$$\begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix} \tag{3.16}$$

instead of diagonal one. This corresponds to the transformation (3.14).

The Poisson brackets corresponding to this structure are defined by the relation

$$\{w, \bar{w}\} = -\frac{\imath}{g}(w - \bar{w})^2, \tag{3.17}$$

while the Killing potentials read

$$h = g\frac{w\bar{w} + 1}{\imath(w - \bar{w})}, \quad h_+ = g\frac{(1 + \imath w)(1 + \imath\bar{w})}{\imath(w - \bar{w})}, \quad h_- = g\frac{(1 - \imath w)(1 - \imath\bar{w})}{\imath(w - \bar{w})} \; . \tag{3.18}$$

Instead of these generators it is more convenient to use their linear combinations

$$H_0 = g\frac{w\bar{w}}{\imath(w - \bar{w})}, \quad K_0 = g\frac{1}{\imath(w - \bar{w})}, \quad D_0 = g\frac{\bar{w} + w}{\imath(w - \bar{w})}, \tag{3.19}$$

$$\{K_0, H_0\} = D_0, \quad \{D_0, H_0\} = 2H_0, \quad \{K_0, D_0\} = 2K_0. \tag{3.20}$$

Here the index 0 indicates that we are dealing with one dimensional case.

Introducing the canonical phase space variables (p, x) [9]

$$w = \frac{p}{x} - \imath\frac{g}{x^2} \quad : \quad \{x, p\} = 1, \tag{3.21}$$

we can represent the Killing potentials in the standard form of the generators of one-dimenisional conformal mechanics [38]

$$H_0 = \frac{p^2}{2} + \frac{g^2}{2x^2}, \quad K_0 = \frac{x^2}{2}, \quad D_0 = px. \tag{3.22}$$

In the next Section we will extend this mapping to the higher-dimensional noncompact complex projective space.

3.3 NONCOMPACT COMPLEX PROJECTIVE SPACE: KLEIN MODEL

Let us construct N-dimensional analog of the Klein model from the Fubini-Study structure of noncompact complex projective space $\widetilde{\mathbb{CP}}^N$ given by the expressions (3.6) with a lower sign.

For this purpose we perform the transformation

$$z^N = \frac{1-\imath w}{1+\imath w}, \quad z^\alpha = \sqrt{2}\frac{\tilde{z}^\alpha}{1+\imath w}, \tag{3.23}$$

which yields the following expressions for the Kähler structure and potential (here and further instead of $\tilde{z}^\alpha$ we use the former notation z^α)

$$ds^2 = \frac{g[dw + \imath\bar{z}^\alpha dz^\alpha][d\bar{w} - \imath z^\beta d\bar{z}^\beta)]}{[\imath(w-\bar{w}) - z^\gamma\bar{z}^\gamma]^2} + \frac{g dz^\alpha d\bar{z}^\alpha}{\imath(w-\bar{w}) - z^\gamma\bar{z}^\gamma}, \tag{3.24}$$

$$\mathcal{K} = -g\log\left[\imath(w-\bar{w}) - z^\gamma\bar{z}^\gamma\right], \qquad \alpha,\beta,\gamma = 1,\ldots N-1, \tag{3.25}$$

with the following range of validity of the coordinates w, z^α

$$\text{Im } w < 0, \qquad \sum_{\alpha=1}^{N-1} z^\alpha\bar{z}^\alpha < -2\text{ Im } w. \tag{3.26}$$

The respective Poisson brackets are defined by the relations

$$\{w,\bar{w}\} = -A(w-\bar{w}), \quad \{w,\bar{z}^\alpha\} = A\bar{z}^\alpha, \quad \{z_\alpha,\bar{z}_\beta\} = \imath A\delta^{\bar{\beta}\alpha}, \tag{3.27}$$

where

$$A := \frac{\imath(w-\bar{w}) - z^\gamma\bar{z}^\gamma}{g}. \tag{3.28}$$

The Killing potentials of the Kähler structure (3.25) are defined by the expressions

$$\begin{aligned}
h_{N\bar{N}} &= \frac{w\bar{w}+1}{A}, \\
h_{\alpha\bar{N}} &= \frac{1}{\sqrt{2}}\frac{\bar{z}^\alpha(1-\imath w)}{A}, \\
h_{\alpha\bar{\beta}} &= \frac{\bar{z}^\alpha z^\beta + \frac{1}{2}\delta_{\alpha\bar{\beta}}(1+\imath w)(1-\imath\bar{w})}{A}, \\
h_N &= \frac{(1+\imath w)(1+\imath\bar{w})}{A}, \\
h_\alpha &= \sqrt{2}\frac{\bar{z}^\alpha(1+\imath w)}{A}.
\end{aligned} \tag{3.29}$$

These potentials form $su(N.1)$ algebra, which in the given notation reads the same as in (3.10) with a lower sign and $a = N, \alpha$. Below we will refer to this representation as the N-dimensional Klein model.

For our purposes, instead of Killing potentials (3.29) it is more convenient to use the following ones

$$H = \frac{w\bar{w}}{A}, \quad K = \frac{1}{A}, \quad D = \frac{w+\bar{w}}{A}, \quad H_{\alpha\bar{N}} = \frac{\bar{z}^{\alpha}w}{A}, \quad H_{\alpha} = \frac{\bar{z}^{\alpha}}{A}, \quad H_{\alpha\bar{\beta}} = \frac{\bar{z}^{\alpha}z^{\beta}}{A}. \tag{3.30}$$

Certainly, these functions are not independent, for there are many obvious relations between them, e.g.

$$H = \frac{1}{2}\sum_{\alpha=1}^{N-1}\frac{H_{\alpha\bar{N}}\bar{H}_{N\bar{\alpha}}}{H_{\alpha\bar{\alpha}}}, \quad H_{\alpha\bar{\beta}} = \frac{H_{\alpha}H_{\bar{\beta}}}{K}, \quad \text{etc.} \tag{3.31}$$

In these terms the $su(1.N)$ algebra relations read

$$\{H,K\} = -D, \quad \{H,D\} = -2H, \quad \{K,D\} = 2K, \tag{3.32}$$

$$\{H,H_{\alpha}\} = -H_{\alpha\bar{N}}, \quad \{H,H_{\alpha\bar{N}}\} = \{H,H_{\alpha\bar{\beta}}\} = 0, \tag{3.33}$$

$$\{K,H_{\alpha\bar{N}}\} = H_{\alpha}, \quad \{K,H_{\alpha}\} = \{K,H_{\alpha\bar{\beta}}\} = 0, \tag{3.34}$$

$$\{D,H_{\alpha}\} = -H_{\alpha}, \quad \{D,H_{\alpha\bar{N}}\} = H_{\alpha\bar{N}}, \quad \{D,H_{\alpha\bar{\beta}}\} = 0, \tag{3.35}$$

$$\{H_{\alpha},H_{\beta}\} = \{H_{\alpha\bar{N}},H_{\beta\bar{N}}\} = \{H_{\alpha},H_{\beta\bar{N}}\} = 0, \tag{3.36}$$

$$\{H_{\alpha},H_{\bar{\beta}}\} = -\imath K\delta_{\alpha\bar{\beta}}, \quad \{H_{\alpha\bar{N}},H_{N\bar{\beta}}\} = -\imath H\delta_{\alpha\bar{\beta}}, \tag{3.37}$$

$$\{H_{\alpha\bar{\beta}},H_{\gamma\bar{\delta}}\} = \imath(H_{\alpha\bar{\delta}}\delta_{\gamma\bar{\beta}} - H_{\gamma\bar{\beta}}\delta_{\alpha\bar{\delta}}), \tag{3.38}$$

$$\{H_{\alpha},H_{N\bar{\beta}}\} = H_{\alpha\bar{\beta}} + \frac{1}{2}\left(g + \sum_{\gamma}H_{\gamma\bar{\gamma}} - \imath D\right)\delta_{\alpha\bar{\beta}}, \tag{3.39}$$

$$\{H_{\alpha},H_{\beta\bar{\gamma}}\} = -\imath H_{\beta}\delta_{\alpha\bar{\gamma}}, \quad \{H_{\alpha\bar{N}},H_{\beta\bar{\gamma}}\} = -\imath H_{\beta\bar{N}}\delta_{\alpha\bar{\gamma}} \tag{3.40}$$

So, the generators H, K, D define the conformal algebra $su(1.1) = so(1.2)$, and the generators $H_{\alpha\bar{\beta}}$ define the algebra $u(N-1)$.

It is seen that

- the Hamiltonian H has two sets of constants of motion $H_{\alpha\bar{N}}$ and $H_{\alpha\bar{\beta}}$ (see (3.33)), therefore it defines superintegrable system;

- the Hamiltonian K has two sets of constants of motion as well, H_{α} and $H_{\alpha\bar{\beta}}$ (see(3.34)). Thus, it defines the superintegrable system as well;

- the triples $(H, H_{\alpha\bar{N}}, H_{\alpha\bar{\beta}})$ and $(K, H_\alpha, H_{\alpha\bar{\beta}})$ transform into each other within discrete transformation

$$(w, z^\alpha) \to (-\frac{1}{w}, \frac{z^\alpha}{w}) \quad \Rightarrow D \to -D, \quad \begin{cases} (H, H_{\alpha\bar{N}}, H_{\alpha\bar{\beta}}) & \to & (K, -H_\alpha, H_{\alpha\bar{\beta}}), \\ (K, H_\alpha, H_{\alpha\bar{\beta}}) & \to & (H, H_{\alpha\bar{N}}, H_{\alpha\bar{\beta}}). \end{cases} \tag{3.41}$$

Adding to the Hamiltonian H the appropriate function of K, we get the superintegrable oscillator- and Coulomb-like systems.

3.3.1 OSCILLATOR-LIKE HAMILTONIAN

We define the oscillator-like Hamiltonian by the expression (cf.(3.22))

$$H_{osc} = H + \omega^2 K \tag{3.42}$$

and introduce the following generators

$$A_\alpha = H_{\alpha\bar{N}} + \imath\omega H_\alpha, \quad B_\alpha = H_{\alpha\bar{N}} - \imath\omega H_\alpha \ : \quad \begin{cases} \{H_{osc}, A_\alpha\} & = & -\imath\omega A_\alpha, \\ \{H_{osc}, B_\alpha\} & = & \imath\omega B_\alpha. \end{cases} \tag{3.43}$$

These generators and their complex conjugates form the following algebra

$$\{A_\alpha, \bar{A}_\beta\} = -\imath\big(H_{osc} - \omega(g + \sum_{\gamma=1}^{N-1} H_{\gamma\bar{\gamma}})\big)\delta_{\alpha\bar{\beta}} + 2\imath\omega H_{\alpha\bar{\beta}}, \tag{3.44}$$

$$\{B_\alpha, \bar{B}_\beta\} = -\imath\big(H_{osc} + \omega(g + \sum_{\gamma=1}^{N-1} H_{\gamma\bar{\gamma}})\big)\delta_{\alpha\bar{\beta}} - 2\imath\omega H_{\alpha\bar{\beta}}, \tag{3.45}$$

$$\{A_\alpha, \bar{B}_\beta\} = -\imath\delta_{\alpha\bar{\beta}}\big(H_{osc} - 2\omega^2 K + \imath\omega D\big), \tag{3.46}$$

with their Poisson brackets with $H_{\alpha\bar{\beta}}$ reading

$$\{A_\alpha, H_{\beta\bar{\gamma}}\} = -\imath\delta_{\alpha\bar{\gamma}} A_\beta, \qquad \{B_\alpha, H_{\beta\bar{\gamma}}\} = -\imath\delta_{\alpha\bar{\gamma}} B_\beta, \tag{3.47}$$

$$\{\bar{A}_\alpha, H_{\beta\bar{\gamma}}\} = -\imath\delta_{\bar{\alpha}\beta}\bar{A}_\gamma, \qquad \{\bar{B}_\alpha, H_{\beta\bar{\gamma}}\} = -\imath\delta_{\bar{\alpha}\beta}\bar{B}_\gamma, \tag{3.48}$$

$$\{\bar{H}_{osc}, H_{\alpha\bar{\beta}}\} = 0. \tag{3.49}$$

Then we immediately deduce that the Hamiltonian (3.42) besides $H_{\alpha\bar{\beta}}$, has the additional constants of motion which provide the system by the maximal superintegrability property

$$
\begin{aligned}
M_{\alpha\beta} &= A_\alpha B_\beta \\
&= H_{\alpha\bar{N}} H_{\beta\bar{N}} + \omega^2 H_\alpha H_\beta + \imath\omega(H_\alpha H_{\beta\bar{N}} - H_{\alpha\bar{N}} H_\beta) \\
&= \frac{\bar{z}^\alpha \bar{z}^\beta}{A^2}(w^2 + \omega^2) \ ,
\end{aligned}
\tag{3.50}
$$

with

$$
\{H_{osc}, M_{\alpha\beta}\} = 0. \tag{3.51}
$$

For sure, these constants of motion are functionally dependent, so that among them one can choose the $N-1$ integrals which guarantee superintegrability of the system, e.g. $M_\alpha \equiv M_{\alpha\alpha}$ only, like in [32]. The generators (3.50) and the $u(N-1)$ generators $H_{\alpha\bar{\beta}}$ form the following symmetry algebra

$$
\{H_{\alpha\bar{\beta}}, M_{\gamma\delta}\} = \imath\delta_{\bar{\beta}\gamma} M_{\alpha\delta} + \imath\delta_{\bar{\beta}\delta} M_{\gamma\alpha}, \qquad \{M_{\alpha\beta}, M_{\gamma\delta}\} = 0, \tag{3.52}
$$

3.3.2 COULOMB-LIKE HAMILTONIAN

We define the Coulomb-like Hamiltonian with the additional constants of motion which provide the system by the maximal superinetgrability property as follows (cf. (3.22))

$$
H_{Coul} = H - \frac{\gamma}{\sqrt{2K}}, \quad R_\alpha = H_{\alpha\bar{N}} + \imath\gamma \frac{H_\alpha}{(g + \sum_{\gamma=1}^{N-1} H_{\gamma\bar{\gamma}})\sqrt{2K}} \ , \tag{3.53}
$$

with

$$
\{H_{Coul}, R_\alpha\} = \{H_{Coul}, H_{\alpha\bar{\beta}}\} = 0. \tag{3.54}
$$

The whole symmetry algebra is as follows

$$\{R_\alpha, R_{\bar\beta}\} = -\imath\delta_{\alpha\bar\beta}\left(H_{Coul} - \frac{\imath\gamma^2}{2(g+\sum_{\gamma=1}^{N-1} H_{\gamma\bar\gamma})^2}\right) + \frac{\imath\gamma^2 H_{\alpha\bar\beta}}{2(g+\sum_{\gamma=1}^{N-1} H_{\gamma\bar\gamma})^3}, \tag{3.55}$$

$$\{H_{\alpha\bar\beta}, R_\gamma\} = \imath\delta_{\gamma\bar\beta}R_\alpha, \tag{3.56}$$

$$\{R_\alpha, R_\beta\} = 0. \tag{3.57}$$

To clarify the origin of these models it is convenient to transit to the canonical coordinates.

3.4 CANONICAL COORDINATES

For the introduction of the canonical coordinates we transit from the complex coordinates to the real ones

$$w = x + \imath y, \quad z^\alpha = q_\alpha \mathrm{e}^{\imath\varphi_\alpha}, \tag{3.58}$$

where

$$y < 0, \quad q_\alpha \geq 0, \quad \varphi_\alpha \in [0, 2\pi), \quad q^2 := \sum_{\alpha=1}^{N-1} q_\alpha^2 < -2y. \tag{3.59}$$

Then we write down the symplectic Kähler one-form and identify it with the canonical one

$$\mathcal{A} = -\frac{g}{2}\frac{dw + d\bar w - \imath(z^\alpha d\bar z^\alpha - \bar z^\alpha dz^\alpha)}{\imath(w-\bar w) - z^\gamma \bar z^\gamma} := p_x dx + \pi_\alpha d\varphi_\alpha. \tag{3.60}$$

This yields the following expressions for the canonical coordinates and momenta,

$$p_x = g\frac{1}{2y+q^2}, \quad \pi_\alpha = -g\frac{q_\alpha^2}{2y+q^2} \quad \Leftrightarrow \quad q_\alpha = \sqrt{-\frac{\pi_\alpha}{p_x}}, \quad y = \frac{\pi+g}{2p_x}, \tag{3.61}$$

where

$$\pi := \sum_{\alpha=1}^{N-1} \pi_\alpha. \tag{3.62}$$

Thus, the complex coordinates are expressed via canonical ones as follows

$$w = x + \imath\frac{\pi+g}{2p_x}, \qquad z^\alpha = \sqrt{-\frac{\pi_\alpha}{p_x}}\mathrm{e}^{\imath\varphi_\alpha}. \tag{3.63}$$

For the complete analogy with one-dimensional case [9] we perform further canonical transformation $(x, p_x) \to (p_r/r, -r^2/2)$ and re-write the above expression in a more convenient form

$$w = \frac{p_r}{r} - \imath\frac{\pi + g}{r^2}, \qquad z^\alpha = \frac{\sqrt{2\pi_\alpha}}{r}\mathrm{e}^{\imath\varphi_\alpha}, \tag{3.64}$$

with

$$r > 0, \quad \pi_\alpha \geq 0, \quad \varphi_\alpha \in [0, 2\pi). \tag{3.65}$$

and

$$A = \frac{\imath(w - \bar{w}) - z^\gamma \bar{z}^\gamma}{g} = \frac{2}{r^2}. \tag{3.66}$$

In these terms the generators of conformal algebra (3.32) take the form of conformal mechanics with separated "radial" and "angular" parts (cf. [36]),

$$H = \frac{p_r^2}{2} + \frac{\mathcal{I}}{r^2}, \quad K = \frac{r^2}{2}, \quad D = p_r r, \tag{3.67}$$

where the angular part of Hamiltonian is given by the expression

$$\mathcal{I} = \frac{1}{2}\left(\sum_{\alpha=1}^{N-1} \pi_\alpha + g\right)^2 . \tag{3.68}$$

The rest generators of $su(1.N)$ algebra read

$$H_{\alpha\bar{N}} = \sqrt{2\pi_\alpha}\left(\frac{p_r}{2} - \imath\frac{\pi + g}{2r}\right)\mathrm{e}^{-\imath\varphi_\alpha}, \tag{3.69}$$

$$H_\alpha = r\sqrt{\frac{\pi_\alpha}{2}}\mathrm{e}^{-\imath\varphi_\alpha}, \tag{3.70}$$

$$H_{\alpha\bar{\beta}} = \sqrt{\pi_\alpha\pi_\beta}\mathrm{e}^{-\imath(\varphi_\alpha - \varphi_\beta)}, \tag{3.71}$$

with the basic Poisson brackets $\{r, p\} = 1$ and $\{\varphi_\alpha, \pi_\alpha\} = 1$.

In these coordinates the oscillator- and Coulomb-like Hamiltonians (3.42),(3.53) take the form,

$$H_{osc} = \frac{p_r^2}{2} + \frac{\mathcal{I}}{r^2} + \frac{\omega^2 r^2}{2}, \quad H_{Coul} = \frac{p_r^2}{2} + \frac{\mathcal{I}}{r^2} - \frac{\gamma}{r}, \tag{3.72}$$

with $\mathcal{I}$ given by (3.68).

The generic conformal mechanics with the angular part $\mathcal{I}_{gen}(\pi, \varphi)$ can be defined via $su(1.N)$ generators by the expression

$$H_{gen} = H + \frac{\mathcal{I}_{gen}(H_\alpha/\sqrt{K}, H_{\bar{\beta}}/\sqrt{K}) - (\sum_{\gamma=1}^{N-1} H_{\gamma\bar{\gamma}} + g)^2}{2K}. \tag{3.73}$$

However, we are mostly interested in the study of integrable and superintegrable systems. Thus, we have to restrict ourselves by the particular cases of angular Hamiltonians.

3.4.1 SUPERINTEGRABLE SYSTEMS

In accordance with Liouville theorem, the integrability of the system with $2N$-dimensional phase space means the existence N functionally independent involutive integrals $F_1 = H, \ldots, F_N$: $\{F_a, F_b\} = 0$. This yields the existence of the so-called action-angle variables $(I_a(F), \Phi_a)$:

$$H = H(I), \quad \{I_a, \Phi_b\} = \delta_{ab}, \quad \{I_a, I_b\} = \{\Phi_a, \Phi_b\} = 0, \tag{3.74}$$

$$\Phi_a \in [0, 2\pi), \qquad a, b = 1, \ldots, N. \tag{3.75}$$

The system becomes maximally superintegrable when the Hamiltonian is expressed via action variables as follows

$$H = H\left(\sum_{a=1}^{N} n_a I_a\right), \quad n_a \in \mathcal{N} \tag{3.76}$$

where n_a are integers (or rational numbers). Indeed, in that case the system possesses the additional (non-involutive) integrals $I_{ab} = \cos(n_a \Phi_b - n_b \Phi_a)$, among them $N-1$ integrals are functionally independent.

Now, let us suppose that $\pi_\alpha, \varphi_\alpha$ are related with the action-angle variables (I_α, Φ_α) of some $(N-1)$-dimensional angular mechanics by the relations

$$\pi_\alpha = n_\alpha I_\alpha, \quad \varphi_\alpha = \frac{\Phi_\alpha}{n_\alpha}, \qquad \text{where} \quad n_\alpha \in \mathcal{N}. \tag{3.77}$$

Upon this identification the angular Hamiltonian (3.68) takes a form

$$\mathcal{I} = \frac{1}{2}\left(\sum_{\alpha=1}^{N-1} n_\alpha I_\alpha + g\right)^2, \qquad \text{with} \quad n_\alpha \in \mathcal{N}, \tag{3.78}$$

This is precisely the class of angular Hamiltonians which provides the superintegrable generalizations of the conformal mechanics, and of the oscillator and Coulomb systems on the N-dimensional Euclidian spaces [56]!

Though the relations (3.32)-(3.40) hold upon this identification, the generators $H_\alpha, H_{\alpha\bar{N}}, H_{\alpha\bar{\beta}}$ become locally defined, $\varphi_\alpha \in [0, 2\pi/m_\alpha)$, so they fail to be constants of motion. However, taking their relevant powers we get the globally defined generators which form the nonlinear algebra

$$\widetilde{H}_\alpha := (H_\alpha)^{n_\alpha} = d_\alpha(I) r^{n_\alpha} \mathrm{e}^{-\imath\Phi_\alpha}, \tag{3.79}$$

$$\widetilde{H}_{\alpha\bar{N}} := (H_{\alpha\bar{N}})^{n_\alpha} = d_{\alpha\bar{N}}(I) \left(p_r - \imath \frac{\sum_{\gamma=1}^{N-1} n_\gamma I_\gamma + g}{r} \right)^{n_\alpha} \mathrm{e}^{-\imath\Phi_\alpha}, \tag{3.80}$$

$$\widetilde{H}_{\alpha\bar{\beta}} := (H_{\alpha\bar{\beta}})^{n_\alpha n_\beta} = d_{\alpha\bar{\beta}}(I) \mathrm{e}^{-\imath\left(n_\beta\Phi_\alpha - n_\alpha\Phi_\beta\right)}, \tag{3.81}$$

where

$$d_\alpha(I) = \left(\frac{n_\alpha I_\alpha}{2}\right)^{n_\alpha/2}, \tag{3.82}$$

$$d_{\alpha\bar{N}}(I) = \left(\frac{n_\alpha I_\alpha}{2}\right)^{n_\alpha/2}, \tag{3.83}$$

$$d_{\alpha\bar{\beta}}(I) = (n_\alpha n_\beta I_\alpha I_\beta)^{n_\alpha n_\beta/2}. \tag{3.84}$$

Thus, we get

$$\{H, \widetilde{H}_{\alpha\bar{N}}\} = \{H, \widetilde{H}_{\alpha\beta}\} = 0, \qquad \{K, \widetilde{H}_\alpha\} = \{K, \widetilde{H}_{\alpha\beta}\} = 0, \tag{3.85}$$

where H, K are defined by (3.67) and (3.78). For sure, the functions (3.84), being dependent on action variables only, do not affect the commutativity of the additional integrals with the Hamiltonian.

In a similar way we construct the constant of motion of the oscillator- and Coulomb-like systems given by (3.42),(3.78) and (3.53),(3.78), respectively.

For the oscillator-like system (3.42) the integrals take the form

$$\widetilde{M}_{\alpha\beta} := (A_\alpha B_\beta)^{n_\alpha n_\beta}$$
$$= \frac{1}{2} d_{\alpha\bar{\beta}}(I) \mathrm{e}^{-\imath\left(n_\beta \Phi_\alpha - n_\alpha \Phi_\beta\right)} \left(\left(\imath p_r + \frac{\sum_{\gamma=1}^{N-1} n_\gamma I_\gamma + g}{r} \right)^2 - \omega^2 r^2 \right)^{n_\alpha n_\beta}, \tag{3.86}$$

with A_α, B_β given by (3.43), (3.77).

For the Coulomb-like system (3.53) the integrals take the form

$$\widetilde{R}_\alpha := (R_\alpha)^{n_\alpha}$$
$$= d_\alpha(I) \mathrm{e}^{-\imath \Phi_\alpha} \left(p_r + \frac{\imath \gamma}{\sum_{\gamma=1}^{N-1} n_\gamma I_\gamma + g} - \frac{\imath \left(\sum_{\gamma=1}^{N-1} n_\gamma I_\gamma + g \right)}{r} \right)^{n_1}. \tag{3.87}$$

There are a few interesting simple systems whose angular parts are given by (3.78) with $g \neq 0$, among them are,

- "charge-monopole" system (and respective systems with oscillator/Coulomb potentials),

$$\mathcal{H} = \sum_{a=1}^{3} \frac{p_a^2}{2} + \frac{s^2}{2r^2}, \tag{3.88}$$

$$\{p_a, x^b\} = \delta_a^b, \quad \{p_a, p_b\} = s\frac{\varepsilon_{abc} x^c}{r^2}, \quad \{x^a, x^b\} = 0. \tag{3.89}$$

  Its angular part is defined by the (two-dimensional) spherical Landau problem, with the following Hamiltonian (see, e.g.[58], where one can find the expressions for action-angle variables for the angular part)

$$\mathcal{I} = \frac{1}{2}\Big(I_1 + I_2 + |s|\Big)^2, \qquad I_{1,2} \in [0, \infty), \tag{3.90}$$

  with s being the monopole number.

- Smorodinsky-Winternitz system

$$H_{SW} = \sum_{a=1}^{N} \Big(\frac{p_a^2}{2} + \frac{g_a^2}{2x_a^2} + \frac{\omega^2 x_a^2}{2} \Big). \tag{3.91}$$

The angular Hamiltonian of this system is given by the expression (3.78) with (see, e.g. [59])

$$k_\alpha = 2\omega, \quad g = \sum_{a=1}^{N-1} |g_a|. \tag{3.92}$$

For sure, this system could be viewed as a trivial case of rational Calogero model, which also belongs to the class of systems above.

- Rational Calogero model associated with Coxeter root system [60] $\mathcal{R} \subset R^N$,

$$\mathcal{H}_{Cal} = \sum_{a=1}^{N} \frac{p_a^2}{2} + \sum_{\alpha\in\mathcal{R}_+} \frac{g_\alpha^2(\alpha\cdot\alpha)}{2(\alpha\cdot x)^2}\ , \qquad \{p_a, x_b\} = \delta_{ab} \tag{3.93}$$

  where $g_\alpha \geq 0$ is a Weyl-invariant multiplicity function on the set of roots [61].

  The spectrum of the angular part of quantum rational Calogero model was found in [62]. Taking its classical limit, one can get the expression of the angular (part of) generalized rational Calogero model in terms of action variables [56]. It given by (3.78), with n_α being the degrees of the basic homogeneous Weyl-invariant polynomials, and $g = \sum_{\alpha\in\mathcal{R}_+} g_\alpha$.

Let us notice that in the angular Hamiltonian (3.68) the nonzero constant $g \neq 0$ appears, and the range of validity of the action variables is fixed to be $I_\alpha \in [0, \infty)$. As a result, the standard oscillator and Coulomb systems cannot be included in the proposed description, since for these systems we should choose $g = 0, I_\alpha \in [0, \infty)$. The first condition leads to the vanishing of Kähler structure and Poisson brackets, while the absorbtion of constant g by the action variables immediately yields the change of the range of validity of the action variables. However, a minor complication allows to involve in our picture the generic superintegrable conformal mechanics, oscillator and Coulomb systems as well.

Using the expressions of the constants of motion presented in [32], we can immediately write down the constants of motions of those systems written in terms of Killing potentials.

- *Conformal mechanics*

$$\mathcal{H} = H - \frac{g(g + 2\sum_{\gamma=1}^{N-1} H_{\gamma\bar{\gamma}})}{4K}, \tag{3.94}$$

$$\mathcal{H}_{\alpha\bar{N}} := H_{\alpha\bar{N}} + \imath\frac{g\bar{z}^{\alpha}}{2} = H_{\alpha\bar{N}} + \imath g\frac{H_\alpha}{2K}\ , \tag{3.95}$$

$$\{\mathcal{H}, \mathcal{H}_{\alpha\bar{N}}\} = \{\mathcal{H}, H_{\alpha\bar{\beta}}\} = 0. \tag{3.96}$$

- *Oscillator-like system*

$$\mathcal{H}_{osc} = H_{osc} - \frac{g(g + 2\sum_{\gamma=1}^{N-1} H_{\gamma\bar{\gamma}})}{4K}, \tag{3.97}$$

$$\{\mathcal{H}_{osc}, H_{\alpha\bar{\beta}}\} = \{\mathcal{H}_{osc}, \mathcal{A}_\alpha\mathcal{B}_\beta\} = 0, \tag{3.98}$$

where

$$\mathcal{A}_\alpha = H_{\alpha\bar{N}} + \imath g\frac{H_\alpha}{2K} + \imath\omega H_\alpha, \quad \mathcal{B}_\alpha = H_{\alpha\bar{N}} + \imath g\frac{H_\alpha}{2K} - \imath\omega H_\alpha \tag{3.99}$$

- *Coulomb-like system*

$$\mathcal{H}_{Coul} = H_{Coul} - \frac{g(g + 2\sum_{\gamma=1}^{N} H_{\gamma\bar{\gamma}})}{4K}, \tag{3.100}$$

$$\{\mathcal{H}_{Coul}, \mathcal{R}_\alpha\} = 0, \tag{3.101}$$

where

$$\mathcal{R}_\alpha = R_\alpha + \imath g\frac{H_\alpha}{\sqrt{2K}}\Big(\frac{1}{\sqrt{2K}} + \frac{\gamma}{(g + \sum_{\gamma=1}^{N} H_{\gamma\bar{\gamma}})\sum_{\gamma=1}^{N} H_{\gamma\bar{\gamma}}}\Big) \tag{3.102}$$

The transition to the action-angle variables (3.77) is obvious.

Hence, we have shown how to describe the superintegrable deformations of oscillator and Coulomb systems in terms of noncompact complex projective spaces $\widetilde{\mathbb{CP}}^N$.

3.5 CONCLUSION

In this chapter, we propose the description of superintegrable models with dynamical $so(1,2)$ symmetry, as well as the generic superintegrable deformations of oscillator and Coulomb systems, in terms of higher-dimensional Klein model (serving as the non-compact analog of complex projective space), considered as the phase space. We provide the expressions for the constants of motion of these systems using Killing potentials that define the $su(N,1)$ isometries of the Kähler structure.

Besides elegant geometric interpretation of (super)integrability of those models, this may be very useful when applying geometric quantization methods.

Chapter 4

$su(1.N|M)$-Superconformal Mechanics and Deformations

4.1 INTRODUCTION

In this chapter we consider the systems with $su(1,N|M)$-symmetric $(N|M)_{\mathbb{C}}$-dimensional Kähler phase space and relate their symmetries with the isometry generators of the super-Kähler structure. We construct this space reducing the $(N+1|M)_{\mathbb{C}}$-dimensional complex pseudo-Euclidian space by the $U(1)$-group action and then identify the reduced phase space with noncompact analog of complex projective superspace constructed in [26]. We parameterize this space by the complex bosonic variable w, $\mathrm{Im}\, w < 0$, by the $N-1$ complex bosonic variables $z^\alpha \in [0,\infty)$, $\arg z \in [0; 2\pi)$, and by M complex fermionic coordinates η^A. Thus, it can be considered as the N-dimensional extension of the Klein model of Lobachevsky plane. This allows us to connect the complex coordinate w with the radial coordinate and momentum of the conformal-mechanical system spanned by $su(1,1)$ subalgebra, and separate the $su(1,1)$ generators interpreting them as Hamiltonian, conformal boosts and dilatation operators. The rest bosonic generators z^α parameterize then the angular part of integrable conformal mechanics with Euclidian configuration spaces. Relating the angular coordinates and momenta with the action-angle variables of the angular part of the integrable conformal mechanics, we describe all symmetries of the generic superintegrable conformal-mechanical systems in terms of the powers of the $su(1,N)$ isometry generators. An important aspect of proposed approach is that in proposed canonical coordinates all fermionic degrees of freedom appear only in the angular part of the Hamiltonian.

Furthermore, we construct the superanalogs of the maximally superintegrable generalizations of the Euclidian oscillator/Coulomb systems considered in [17] as follows: we preserve the form of Hamiltonian expressed via generators of $su(1,1)$ subalgebra but extend the phase space $\widetilde{\mathbb{CP}}^N$ to phase superspace $\widetilde{\mathbb{CP}}^{N|M}$. As a result, we find that these superextensions reserves all symmetries of the initial bosonic Hamiltonians and get maximal set of functionally-independent fermionic integrals,i.e. they remains superintegrable in the sence of super-Liuvville theorem. We also find, that the constructed oscillator-like systems (in contrast with Coulomb-like ones) possess deformed $\mathcal{N} = 2M$ Poincaré supersymmetry, and express all the symmetries of these superintegrable systems via $su(1,N)$ isometry generators as well.

The chapter is organized as follows.

In *Section 4.2* we present the basic facts on Kähler supermanifolds and construct, by the Hamiltonian reduction, the non-compact complex projective superspace $\widetilde{\mathbb{CP}}^{N|M}$ in the parametrization similar to those of Klein model for Lobachevsky space.

In *Section 4.3* we analyze the symmetry algebra of $\widetilde{\mathbb{CP}}^{N|M}$ and extract from it the $su(1,N|M)$-superconformal systems.

In *Section 4.4* we introduce the canonical coordinates which naturally split radial and angular parts of the Hamiltonian and relate the angular part with the systems formulating in terms of action-angle variables. In the *Section 4.5* we construct superintegrable supergeneralizations of oscillator- and Coulomb-like systems.

4.2 NONCOMPACT COMPLEX PROJECTIVE SUPERSPACE

The (even) $(N|M)$-dimensional Kähler supermanifold is defined as a complex symplectic supermanifold whose symplectic structure is given by the expression

$$\Omega = \imath(-1)^{p_I(p_J+1)} g_{I\bar{J}} dZ^I \wedge d\bar{Z}^J, \quad d\Omega = 0, \tag{4.1}$$

with Z^I denoting N complex bosonic coordinates and M complex fermionic ones, while $p_I := p(Z^I)$ be Grassmanian parity of coordinates, i.e. it is equal to zero for bosonic coordinate and to one for the fermionic one.

The "metrics components" $g_{I\bar{J}}$ can then be locally represented in the form

$$g_{I\bar{J}} = \frac{\partial^L}{\partial Z^I} \frac{\partial^R}{\partial \bar{Z}^J} K(Z, \bar{Z}), \tag{4.2}$$

where $\partial^{L(R)}/\partial Z^I$ denotes left(right) derivatives.

The Poisson brackets associated with this Kähler structure looks as follows

$$\{f, g\} = \imath \left(\frac{\partial^R f}{\partial \bar{Z}^I} g^{\bar{I}J} \frac{\partial^L g}{\partial Z^J} - (-1)^{p_I p_J} \frac{\partial^R f}{\partial Z^I} g^{\bar{J}I} \frac{\partial^L g}{\partial \bar{Z}^J} \right), \tag{4.3}$$

where

$$g^{\bar{I}J} g_{J\bar{K}} = \delta^{\bar{I}}_{\bar{K}} \quad \overline{g^{\bar{I}J}} = (-1)^{p_I p_J} g^{\bar{J}I}, \tag{4.4}$$

As in the pure bosonic case, the isometries of Kähler manifolds are given by the *holomorphic Hamiltonian vector fields*,

$$\mathbf{V}_\mu := \{h_\mu(Z, \bar{Z}), \quad \} = V^I(Z) \frac{\partial^L}{\partial Z^I} + \bar{V}^I(\bar{Z}) \frac{\partial^L}{\partial \bar{Z}^I}, \tag{4.5}$$

where $h_\mu(Z, \bar{Z})$ are real functions called Killing potentials (see, e.g. [10, 26] for the details).

Our goal is to study the systems on the Kähler phase space with $su(1,N|M)$ isometry superalgebra. For the construction of such phase space it is convenient, at first, to present the linear realization of $u(N.1|M)$ superconformal algebra on the complex pseudo-Euclidian superspace $\mathbb{C}^{N.1|M}$ equipped with the canonical Kähler structure (and thus, by the canonical supersymplectic structure)and then reduced it by the action of $U(1)$ generator.

It is instructive to present this reduction in details. Let us equip, at first, the $(N+1|M)$-dimensional complex superspace with the canonical symplectic structure

$$\Omega_0 = \imath \sum_{a,b=0}^{N} \gamma_{a\bar{b}} dv^a \wedge d\bar{v}^b + \sum_{A=1}^{M} d\eta^A \wedge d\bar{\eta}^A, \tag{4.6}$$

with $v^a, \bar{v}^a$ being bosonic variables, and $\eta^A, \bar{\eta}^A$ being fermionic ones, and with the matrix $\gamma_{a\bar{b}}$ chosen in the form

$$\gamma_{a\bar{b}} = \left(\begin{array}{cc|ccc} 0 & -i & & & \\ i & 0 & & & \\ \hline & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{array}\right), \qquad a,b = N,0,1,...,N-1. \tag{4.7}$$

With this supersymplectic structure we can associate the Poisson brackets given by the relations

$$\{v^a, \bar{v}^b\} = -\imath\gamma^{\bar{b}a}, \quad \{\eta^A, \bar{\eta}^B\} = \{\bar{\eta}^B, \eta^A\} = \delta^{A\bar{B}}, \qquad \gamma^{\bar{a}b}\gamma_{b\bar{c}} = \delta^{\bar{a}}_{\bar{c}}. \tag{4.8}$$

Equivalently,

$$\{v^0, \bar{v}^N\} = 1, \quad \{v^N, \bar{v}^0\} = -1, \quad \{v^\alpha, \bar{v}^\beta\} = \imath\delta^{\alpha\bar{\beta}}, \tag{4.9}$$

$$\{\eta^A, \bar{\eta}^B\} = \{\bar{\eta}^B, \eta^A\} = \delta^{A\bar{B}}. \tag{4.10}$$

Here we introduced the indices $\alpha, \beta = 1, \ldots, N-1$.

On this superspace we can define the linear Hamiltonian action of $u(N.1|M) = u(1) \times su(N.1|M)$ superalgebra

$$\{h_{a\bar{b}}, h_{c\bar{d}}\} = -\imath\left(h_{a\bar{d}}\gamma^{\bar{c}b} - h_{c\bar{b}}\gamma^{\bar{a}d}\right), \tag{4.11}$$

$$\{\Theta_{A\bar{a}}, \bar{\Theta}_{\bar{B}b}\} = h_{b\bar{a}}\delta^{B\bar{A}} - R_{A\bar{B}}\gamma^{\bar{b}a}, \quad \{\Theta_{A\bar{a}}, h_{b\bar{c}}\} = -\imath\Theta_{A\bar{c}}\gamma^{\bar{b}a}, \tag{4.12}$$

$$\{R_{A\bar{B}}, R_{C\bar{D}}\} = \imath\left(R_{A\bar{D}}\delta^{B\bar{C}} - R_{C\bar{B}}\delta^{D\bar{A}}\right), \quad \{\Theta_{A\bar{a}}, R_{C\bar{D}}\} = -\imath\Theta_{C\bar{a}}\delta^{D\bar{A}}, \tag{4.13}$$

where

$$h_{a\bar{b}} = \upsilon^a \upsilon^b, \qquad \Theta_{A\bar{a}} = \bar{\eta}^A \upsilon^a, \qquad R_{A\bar{B}} = \imath\bar{\eta}^A \eta^B. \tag{4.14}$$

The $u(1)$ generator defining the center of $u(N.1|M)$ is given by the expression

$$J = \gamma_{a\bar{b}} \upsilon^a \bar{\upsilon}^b + \imath\eta^A \bar{\eta}^A \ : \quad \{J, h_{a\bar{b}}\} = \{J, \Theta_{A\bar{a}}\} = \{J, R_{A\bar{B}}\} = 0. \tag{4.15}$$

Hence, reducing the system by the action of this generator we will get the "non-compact" projective super-space $\widetilde{\mathbb{CP}}^{N|M}$ (i.e. the supergeneralization of noncompact projective space $\widetilde{\mathbb{CP}}^{N}$), which is $(2N|2M)$-(real)dimensional space.

For performing the reduction by the action of generator (4.15) we have to choose, at first, the $2N$ real (N complex) bosonic and $2M$ real (N complex) fermionic functions commuting with J, and . Then, we have to calculate their Poisson brackets and restrict the latter to the level surface

$$J = g. \tag{4.16}$$

As a result we will get the Poisson brackets on the reduced $(2N|2M)$-(real) dimensional space, with that $U(1)$-invariant functions playing the role of the latter's coordinates.

The required functions could be easily found,

$$w = \frac{\upsilon^N}{\upsilon^0}, \quad z^\alpha = \frac{\upsilon^\alpha}{\upsilon^0}, \quad \theta^A = \frac{\eta^A}{\upsilon^0} \ : \quad \{w, J\} = \{z^a, J\} = \{\theta^A, J\} = 0, \quad \text{and} \quad c.c.. \tag{4.17}$$

Calculating their Poisson brackets and having in mind the expression following from (4.16),

$$A := \left.\frac{1}{\upsilon^0 \bar{\upsilon}^0}\right|_{J=g} = \frac{1}{g}\left(\imath(w - \bar{w}) - \sum_{\gamma=1}^{N-1} z^\gamma \bar{z}^\gamma + \imath \sum_{C=1}^{M} \theta^C \bar{\theta}^C\right), \tag{4.18}$$

we get the reduced Poisson brackets defined by the following non-zero relations (and their complex conjugates)

$$\{w, \bar{w}\} = -A(w - \bar{w}), \tag{4.19}$$

$$\{z^\alpha, \bar{z}^\beta\} = \imath A \delta^{\alpha\bar{\beta}}, \tag{4.20}$$

$$\{\theta^A, \bar{\theta}^B\} = A \delta^{A\bar{B}}, \tag{4.21}$$

$$\{w, \bar{z}^\alpha\} = A\bar{z}^\alpha, \tag{4.22}$$

$$\{w, \bar{\theta}^A\} = A\bar{\theta}^A. \tag{4.23}$$

These Poisson brackets are associated with the supersymplectic structure

$$\Omega = \frac{\imath}{g}\left[\frac{1}{A^2}dw\wedge d\bar{w} - \frac{\imath z^\alpha}{A^2}dw\wedge d\bar{z}^\alpha - \frac{\theta^A}{A^2}dw\wedge d\bar{\theta}^A\right.$$
$$+\frac{\imath\bar{z}^\alpha}{A^2}dz^\alpha\wedge d\bar{w} + \left(\frac{g\delta_{\alpha\bar{\beta}}}{A}+\frac{\bar{z}^\alpha z^\beta}{A^2}\right)dz^\alpha\wedge d\bar{z}^\beta - \frac{\imath\bar{z}^\alpha\theta^A}{A^2}dz^\alpha\wedge d\bar{\theta}^A$$
$$\left.-\frac{\bar{\theta}^A}{A^2}d\theta^A\wedge d\bar{w} + \frac{\imath\bar{\theta}^A z^\alpha}{A^2}d\theta^A\wedge d\bar{z}^\alpha - \left(\frac{\imath g\delta_{A\bar{B}}}{A}+\frac{\bar{\theta}^A\theta^B}{A^2}\right)d\theta^A\wedge d\bar{\theta}^B\right]. \qquad (4.24)$$

It is defined by the Kähler potential

$$\mathcal{K} = -g\log(\imath(w-\bar{w}) - z^\alpha\bar{z}^\alpha + \imath\theta^A\bar{\theta}^A). \qquad (4.25)$$

In what follows we will call this space " noncompact projective superspace $\widetilde{\mathbb{CP}}^{N|M}$ ". The isometry algebra of this space is $su(N,1|M)$, which can be easily obtained by the restriction of (4.12),(4.13) to the level surface (4.16). It is defined by the following Killing potentials

$$H := v^N\bar{v}^N|_{J=g} = \frac{w\bar{w}}{A},$$
$$K := v^0\bar{v}^0|_{J=g} = \frac{1}{A},$$
$$D := (v^N\bar{v}^0 + v^0\bar{v}^N)|_{J=g} = \frac{w+\bar{w}}{A}, \qquad (4.26)$$
$$H_\alpha := \bar{v}^\alpha v^N|_{J=g} = \frac{\bar{z}^\alpha w}{A},$$
$$K_\alpha := \bar{v}^\alpha v^0|_{J=g} = \frac{\bar{z}^\alpha}{A},$$
$$h_{\alpha\bar{\beta}} := \bar{v}^\alpha v^\beta|_{J=g} = \frac{\bar{z}^\alpha z^\beta}{A}, \qquad (4.27)$$
$$Q_A := \bar{\eta}^A v^N|_{J=g} = \frac{\bar{\theta}^A w}{A},$$
$$S_A := \bar{\eta}^A v^0|_{J=g} = \frac{\bar{\theta}^A}{A},$$
$$\Theta_{A\bar{\alpha}} := \bar{\eta}^A v^\alpha|_{J=g} = \frac{\bar{\theta}^A z^\alpha}{A}, \qquad (4.28)$$
$$R_{A\bar{B}} := \imath\bar{\eta}^A\eta^B|_{J=g} = \imath\frac{\bar{\theta}^A\theta^B}{A}. \qquad (4.29)$$

Constructed super-Kähler structure can be viewed as a higher dimensional analog of the Klein model of Lobachevsky space, where the latter is parameterized by the lower half-plane. One can choose, instead of non-diagonal matrix (4.7), the diagonal one, $\gamma_{a\bar{b}} = diag(1,-1,\ldots,-1)$. In that case the reduced Kähler structure will have the Fubini-Study-like form (see Section VI).

In the next Section we will analize the isometry algebra defined by these generators in details. Presented choice (4.7) is motivated by the by its convenience for the analizing superconformal mechanics. Indeed, in thet case the generators (4.26) defines conformal subalgebra $su(1.1)$ and are separated wrom the rest $su(N,1)$ generators. Thus they can be interpreted as the Hamiltonian of conformal mechanics, the generator of conformal boosts and the generator of dilatation.

In the next section we will analize in details these superconformal mechanics and their dynamical super algebra the isometry algebra defined by the generators (4.26),(4.27),(4.28),(4.29).

4.3 $su(1,N|M)$ SUPERCONFORMAL ALGEBRA

The generators (Killing potentials) (4.26),(4.27),(4.28),(4.29) form $su(1,N|M)$ superalgebra given by (4.12),(4.13) with $\gamma_{a\bar{b}}$ defined in (4.7). Its explicit expression with separated $su(1,1)$ subalgebra is represented below. For the convenience it is divided into three sectors: "bosonic", "fermionic" and "mixed" ones.

BOSONIC SECTOR: $su(1,N) \times u(M)$

The bosonic sector is direct product of the $su(1,N)$ algebra defined by the generators (4.26),(4.27), and the $u(M)$ algebra defined by the R-symmetry generators (4.29). Explicitly, the $su(1.N)$ algebra is given by the relations

$$\{H, K\} = -D, \quad \{H, D\} = -2H, \quad \{K, D\} = 2K, \tag{4.30}$$

$$\{H, K_\alpha\} = -H_\alpha, \quad \{H, H_\alpha\} = \{H, h_{\alpha\bar{\beta}}\} = 0, \tag{4.31}$$

$$\{K, H_\alpha\} = K_\alpha, \quad \{K, K_\alpha\} = \{K, h_{\alpha\bar{\beta}}\} = 0, \tag{4.32}$$

$$\{D, K_\alpha\} = -K_\alpha, \quad \{D, H_\alpha\} = H_\alpha, \quad \{D, h_{\alpha\bar{\beta}}\} = 0, \tag{4.33}$$

$$\{K_\alpha, K_\beta\} = \{H_\alpha, H_\beta\} = \{K_\alpha, H_\beta\} = 0, \tag{4.34}$$

$$\{K_\alpha, \overline{K}_\beta\} = -\imath K \delta_{\alpha\bar{\beta}}, \quad \{H_\alpha, \overline{H}_\beta\} = -\imath H \delta_{\alpha\bar{\beta}}, \tag{4.35}$$

$$\{h_{\alpha\bar{\beta}}, h_{\gamma\bar{\delta}}\} = \imath(h_{\alpha\bar{\delta}}\delta_{\gamma\bar{\beta}} - h_{\gamma\bar{\beta}}\delta_{\alpha\bar{\delta}}), \tag{4.36}$$

$$\{K_\alpha, h_{\beta\bar{\gamma}}\} = -\imath K_\beta \delta_{\alpha\bar{\gamma}}, \quad \{H_\alpha, h_{\beta\bar{\gamma}}\} = -\imath H_\beta \delta_{\alpha\bar{\gamma}}, \tag{4.37}$$

$$\{K_\alpha, \overline{H}_\beta\} = h_{\alpha\bar{\beta}} + \frac{1}{2}\left(I - \imath D\right)\delta_{\alpha\bar{\beta}}, \tag{4.38}$$

where

$$I := g + \sum_{\gamma=1}^{N-1} h_{\gamma\bar{\gamma}} + \sum_{C=1}^{M} R_{C\bar{C}} \tag{4.39}$$

The R-symmetry generators form $u(M)$ algebra and commute with all generators of $su(1,N)$:

$$\{R_{A\bar{B}}, R_{C\bar{D}}\} = \imath(R_{A\bar{D}}\delta_{C\bar{B}} - R_{C\bar{B}}\delta_{A\bar{D}}), \quad \{R_{A\bar{B}}, (H; K; D; K_\alpha; H_\alpha; h_{\alpha\bar{\beta}})\} = 0. \tag{4.40}$$

It is clear that the generators H, D, K form conformal algebra $su(1,1)$, the generators $h_{\alpha\bar{\beta}}$ form the algebra $u(N-1)$, and all together - the $su(1,1)\times(N-1)$ algebra. Notice, that the generator I in (4.39) defines the Casimir of conformal algebra $su(1,1)$:

$$\mathcal{I} := \frac{1}{2}I^2 = \frac{1}{2}D^2 - 2HK. \tag{4.41}$$

Hence, choosing H to the role of Hamiltonian, we get that H_α, $h_{\alpha\bar{\beta}}, R_{A\bar{B}}$ define its constant of motion. Similarly, choosing to the role of Hamiltonian the generator K, we get that it has constants of motion $K_\alpha, h_{\alpha\bar{\beta}}, R_{A\bar{B}}$.

"FERMIONIC" SECTOR

The Poisson brackets between fermionic generators (4.28) are as follows

$$\{S_A, \overline{S}_B\} = K\delta_{A\bar{B}}, \quad \{Q_A, \overline{Q}_B\} = H\delta_{A\bar{B}}, \tag{4.42}$$

$$\{S_A, \overline{Q}_B\} = -\imath R_{A\bar{B}} + \frac{\imath}{2}\left(I - \imath D\right)\delta_{A\bar{B}}, \tag{4.43}$$

$$\{\Theta_{A\bar{\alpha}}, \overline{\Theta}_{B\bar{\beta}}\} = R_{A\bar{B}}\delta_{\beta\bar{\alpha}} + h_{\beta\bar{\alpha}}\delta_{A\bar{B}}, \tag{4.44}$$

$$\{S_A, \overline{\Theta}_{B\bar{\alpha}}\} = K_\alpha \delta_{A\bar{B}}, \quad \{Q_A, \overline{\Theta}_{B\bar{\alpha}}\} = H_\alpha \delta_{A\bar{B}}, \tag{4.45}$$

$$\{S_A, S_B\} = \{Q_A, Q_B\} = \{\Theta_{A\bar{\alpha}}, \Theta_{B\bar{\beta}}\} = \{S_A, Q_B\} = \{S_A, \Theta_{B\bar{\alpha}}\} = \{Q_A, \Theta_{B\bar{\alpha}}\} = 0. \tag{4.46}$$

Hence, the functions Q_A play the role of supercharges for the Hamiltonian H, and the functions S_A define the supercharges of the Hamiltonian K playing the role of generator of conformal boosts.

"MIXED" SECTOR

The mixed sector is given by the relations

$$\{H, Q_A\} = \{H, \Theta_{A\bar{\alpha}}\} = 0, \quad \{H, S_A\} = -Q_A, \tag{4.47}$$

$$\{K, S_A\} = \{K, \Theta_{A\bar{\alpha}}\} = 0, \quad \{K, Q_A\} = S_A, \tag{4.48}$$

$$\{D, S_A\} = -S_A, \quad \{D, Q_A\} = Q_A, \quad \{D, \Theta_{A\bar{\alpha}}\} = 0 \tag{4.49}$$

$$\{Q_A, \overline{K}_\alpha\} = -\Theta_{A\bar{\alpha}}, \tag{4.50}$$

$$\{Q_A, H_\alpha\} = \{Q_A, \bar{H}_\alpha\} = \{Q_A, \bar{K}_\alpha\} = \{Q_A, h_{\alpha\bar{\beta}}\} = 0, \tag{4.51}$$

$$\{S_A, \overline{H}_\alpha\} = \Theta_{A\bar{\alpha}}, \tag{4.52}$$

$$\{S_A, K_\alpha\} = \{S_A, \bar{K}_\alpha\} = \{S_A, H_\alpha\} = \{S_A, h_{\alpha\bar{\beta}}\} = 0, \tag{4.53}$$

$$\{\Theta_{A\bar{\alpha}}, K_\beta\} = \imath S_A \delta_{\beta\bar{\alpha}}, \quad \{\Theta_{A\bar{\alpha}}, H_\beta\} = \imath Q_A \delta_{\beta\bar{\alpha}}, \quad \{\Theta_{A\bar{\alpha}}, h_{\beta\bar{\gamma}}\} = \imath \Theta_{A\bar{\gamma}} \delta_{\beta\bar{\alpha}}, \tag{4.54}$$

$$\{\Theta_{A\bar{\alpha}}, \bar{H}_\alpha\} = \{\Theta_{A\bar{\alpha}}, \bar{K}_\alpha\} = 0, \tag{4.55}$$

$$\{S_A, R_{B\bar{C}}\} = -\imath S_B \delta_{A\bar{C}}, \quad \{Q_A, R_{B\bar{C}}\} = -\imath Q_B \delta_{A\bar{C}}, \quad \{\Theta_{A\bar{\alpha}}, R_{B\bar{C}}\} = -\imath \Theta_{B\bar{\alpha}} \delta_{A\bar{C}}. \tag{4.56}$$

Looking to the all Poisson bracket relations together we conclude that

- The bosonic functions H_α, $h_{\alpha\bar{\beta}}$, and the fermionic functions Q_A, $\Theta_{A\bar{\alpha}}$ commute with the Hamitonian H and thus, provide it by the superintegrability property [1];
- The bosonic functions K_α, $h_{\alpha\bar{\beta}}$ and the fermionic functions S_A,$\Theta_{A\bar{\alpha}}$ commute with the generator K. Hence, the Hamiltonian K defines the superintegrable system as well.
- The triples $(H, H_\alpha, Q_A,)$ and $(K, K_\alpha, S_A,)$ transform into each other under the discrete transformation

$$(w, z^\alpha, \theta^A) \to (-\frac{1}{w}, \frac{z^\alpha}{w}, \frac{\theta^A}{w}) \quad \Rightarrow \tag{4.57}$$

[1] In accord with superanalog of Liuville theorem [68] the system on $(2N.M)$ phase superspace is integrable iff it possess N commuting bosonic integrals (with nonvanishing and functionally independent bosonic parts) and M fermionic ones

$$D \to -D, \quad \begin{cases} (H, H_\alpha, Q_A,) & \to \quad (K, -K_\alpha, -S_A), \\ (K, K_\alpha, S_A) & \to \quad (H, H_\alpha, Q_A,) \end{cases} . \tag{4.58}$$

- The functions $h_{\alpha\bar\beta}, \Theta_{A\bar\alpha}$ are invariant under discrete transformation (4.58).Moreover, they appear to be constants of motion both for H and K. Hence, they remain to be constants of motion for any Hamiltonian being the functions of H, K. In particular, adding to the Hamiltonian H the appropriate function of K, we get the superintegrable oscillator- and Coulomb-like systems with dynamical superconformal symmetry .

- The superalgebra $su(1, N|M)$ admits 5-graded decomposition

$$su(1, N|M) = \mathfrak{f}_{-2} \oplus \mathfrak{f}_{-1} \oplus \mathfrak{f}_0 \oplus \mathfrak{f}_{+1} \oplus \mathfrak{f}_{+2} \tag{4.59}$$

with

$$[\mathfrak{f}_i, \mathfrak{f}_j] \subseteq \mathfrak{f}_{i+j} \quad \text{for } i, j \in \{-2, -1, 0, 1, 2\}\,, \tag{4.60}$$

where $\mathfrak{f}_i = 0$ for $|i| > 2$ is understood. The subset $\mathfrak{f}_0$ includes the generators $D, h_{\alpha\bar\beta}, \Theta_{A\bar\alpha}, \overline{\Theta}A\bar\alpha, R_{A\bar B}$, the subsets $\mathfrak{f}_{-2}$ and $\mathfrak{f}_2$ contain only generators H and K, respectively, while the subsets $\mathfrak{f}_{-1}$ and $\mathfrak{f}_1$ contain the generators $H_\alpha, \bar H_\alpha, Q_A, \bar Q_A$ and $K_\alpha, \bar K_\alpha, S_A, \bar S_A$.

Let us conclude this section by the following remark. It is easy to see, that the generator (4.39) commutes the generators $H, D, K, S_A, Q_A, R_{A\bar B}$. Hence, these generators form superconformal algebra $su(1.1|M)$ with central charge $\sqrt{2\mathcal{I}}$ (4.41) (being the casimir of $su(1, 1|M)$) as well)

$$\{H, K\} = -D, \quad \{H, D\} = -2H, \quad \{K, D\} = 2K, \tag{4.61}$$

$$\{S_A, \overline{S}_B\} = K\delta_{A\bar B}, \quad \{Q_A, \overline{Q}_B\} = H\delta_{A\bar B},$$

$$\{S_A, \overline{Q}_B\} = -\imath R_{A\bar B} + \frac{\imath}{2}\left(\sqrt{2\mathcal{I}} - \imath D\right)\delta_{A\bar B},$$

$$\{H, S_A\} = -Q_A, \quad \{K, Q_A\} = S_A, \quad \{H, Q_A\} = \{K, S_A\} = 0, \tag{4.62}$$

$$\{D, S_A\} = -S_A, \quad \{D, Q_A\} = Q_A,$$

$$\{R_{A\bar B}, R_{C\bar D}\} = \imath(R_{A\bar D}\delta_{C\bar B} - R_{C\bar B}\delta_{A\bar D}), \quad \{S_A, R_{B\bar C}\} = -\imath S_B\delta_{A\bar C}, \tag{4.63}$$

$$\{Q_A, R_{B\bar C}\} = -\imath Q_B\delta_{A\bar C}. \tag{4.64}$$

In the next section we will express presented $su(1, N|M)$ generators in appropriate canonical coordinates and in this way we will relate presented formulae with the superextensions of conventional conformal mechanics.

4.4 CANONICAL COORDINATES AND ACTION-ANGLE VARIABLES

For the introduction of the canonical coordinates we transit from the complex coordinates to the real ones for bosonic variables and make a change of fermionic ones such that the new fermionic variables will have canonical Poisson brackets. For this purpose we represent bosonic variables w, z^α as follows,

$$w = x + \imath y, \quad z^\alpha = q_\alpha \mathrm{e}^{\imath\varphi_\alpha}, \tag{4.65}$$

where

$$y < 0, \quad q_\alpha \geq 0, \quad \varphi_\alpha \in [0, 2\pi), \quad q^2 := \sum_{\alpha=1}^{N-1} q_\alpha^2 < -2y. \tag{4.66}$$

Then we write down the symplectic/Kähler one-form and identify it with the canonical one

$$\begin{aligned}\mathcal{A} = &-\frac{g}{2}\frac{dw + d\bar{w} - \imath(z^\alpha d\bar{z}^\alpha - \bar{z}^\alpha dz^\alpha) + \theta^A d\bar{\theta}^A + \bar{\theta}^A d\theta^A}{\imath(w - \bar{w}) - z^\gamma \bar{z}^\gamma + \imath\theta^C\bar{\theta}^C} \\ &:= p_x dx + \pi_\alpha d\varphi_\alpha + \frac{1}{2}\chi^A d\bar{\chi}^A + \frac{1}{2}\bar{\chi}^A d\chi^A.\end{aligned} \tag{4.67}$$

After some calculations and canonical transformation $(p_x, x) \to (-r^2/2, p_r/r)$, one can obtain

$$w = \frac{p_r}{r} - \imath\frac{I}{r^2}, \quad z^\alpha = \frac{\sqrt{2\pi_\alpha}}{r}\mathrm{e}^{\imath\varphi_\alpha}, \quad \theta^A = \frac{\sqrt{2}}{r}\chi^A, \tag{4.68}$$

with

$$\{r, p_r\} = 1, \quad \{\varphi_\beta, \pi_\alpha\} = \delta_{\alpha\beta}, \quad \{\chi^A, \bar{\chi}^B\} = \delta^{A\bar{B}}, \tag{4.69}$$

$$\pi_a \geq 0, \quad \varphi^a \in [0, 2\pi), \quad r > 0. \tag{4.70}$$

Hence, $r, p_r, \varphi^\alpha, \pi_\alpha, \chi^A, \bar\chi^A$ define canonical coordinates. They expresses via initial ones as follows

$$p_r = \frac{w+\bar w}{2}\sqrt{\frac{2}{A}}, \quad r = \sqrt{\frac{2}{A}}, \tag{4.71}$$
$$\pi_\alpha = \frac{z^\alpha \bar z^\alpha}{A}, \quad \varphi_\alpha = \arg(z^\alpha), \tag{4.72}$$
$$\chi^A = -\frac{\theta^A}{\sqrt{A}}, \quad \bar\chi^A = -\frac{\bar\theta^A}{\sqrt{A}}, \tag{4.73}$$

where

$$I = g + \sum_{\alpha=1}^{N-1}\pi_\alpha + \sum_{A=1}^{M}\imath\bar\chi^A\chi^A\ , \qquad A := \frac{\imath(w-\bar w) - z^\gamma\bar z^\gamma + \imath\theta^C\bar\theta^C}{g} = \frac{2}{r^2}. \tag{4.74}$$

In these canonical coordinates the isometry generators read

$$H = \frac{p_r^2}{2} + \frac{I^2}{2r^2}, \quad K = \frac{r^2}{2}, \quad D = p_r r, \tag{4.75}$$
$$H_\alpha = \sqrt{\frac{\pi_\alpha}{2}}\mathrm{e}^{-\imath\varphi_\alpha}\left(p_r - \imath\frac{I}{r}\right), \quad K_\alpha = r\sqrt{\frac{\pi_\alpha}{2}}\mathrm{e}^{-\imath\varphi_\alpha}, \quad h_{\alpha\bar\beta} = \sqrt{\pi_\alpha\pi_\beta}\mathrm{e}^{-\imath(\varphi_\alpha-\varphi_\beta)}, \tag{4.76}$$
$$Q_A = \frac{\bar\chi^A}{\sqrt{2}}\left(p_r - \imath\frac{\sqrt{2\mathcal{I}}}{r}\right), \quad S_A = \frac{\bar\chi^A}{\sqrt{2}}r, \quad \Theta_{A\bar\alpha} = \bar\chi^A\sqrt{\pi_\alpha}\mathrm{e}^{\imath\varphi_\alpha}, \tag{4.77}$$
$$R_{A\bar B} = \imath\bar\chi^A\chi^B. \tag{4.78}$$

Interpreting r as a radial coordinate, and p_r as radial momentum, we get the superconformal mechanics with angular Hamiltonian given by

$$\mathcal{I} := \frac{I_0 + (\bar\chi\chi)}{2}, \qquad \text{with} \qquad I_0 := g + \sum_{\alpha=1}^{N-1}\pi_\alpha, \quad (\bar\chi\chi) := \sum_{A=1}^{M}\imath\bar\chi^A\chi^A\ . \tag{4.79}$$

So, the fermionic part of superconformal Hamiltonian is encoded in its angular part.

The explicit dependence of Hamiltonian H and of its supercharges Q_A and on fermions is as follows

$$H = H_0 + \frac{I_0(\bar\chi\chi)}{r^2} + \frac{(\bar\chi\chi)^2}{2r^2}, \qquad Q_A = -\frac{\bar\chi^A}{\sqrt{2}}\left(p_r - \imath\frac{I_0}{r} - \imath\frac{(\bar\chi\chi)}{r}\right), \tag{4.80}$$

while the dependence of bosonic integrals H_α on fermions is given by the expression

$$H_\alpha = H_\alpha^0 - \frac{K_\alpha(\bar\chi\chi)}{2K}, \tag{4.81}$$

where

$$H_0 := \frac{p_r^2}{2} + \frac{I_0^2}{2r^2}, \quad H^0_\alpha = \sqrt{\frac{\pi_\alpha}{2}} \mathrm{e}^{-\imath\varphi_\alpha} \left(p_r - \imath \frac{I_0}{r} \right) \quad : \quad \{H^0_\alpha, H^0\} = 0. \tag{4.82}$$

So, proposed superconformal Hamiltonian H inherits all symmetries of initial Hamiltonian H_0 (given by $H^0_\alpha, h_{\alpha\bar{\beta}}$).

Looking at the functional dependence of the angular Hamiltonian $\mathcal{I}$ from the angular variables $\varphi^\alpha, \pi_\alpha$ one can expect that the set of conformal mechanics admitting proposed $su(1, N|M)$ superconformal extensions seems very restricted. However, it is not a case, since we it is not necessary to interpret φ^α as a coordinate of the configuration space, and π_α as its canonically conjugated momentum. Instead, since π_α define a constant of motion of the bosonic Hamiltonian H_0 (and of the respective angular Hamiltonian $\mathcal{I}_0 = H_0 K/2 - D^2$), we can interpret it as the action variable I_α, and consider φ^α as a respective angle variable Φ_α .

Furthermore, suppose that $\pi_\alpha, \varphi_\alpha$ are related with the action-angle variables (I_α, Φ_α) of some $(N-1)$-dimensional angular mechanics by the relations

$$\pi_\alpha = n_\alpha I_\alpha, \quad \varphi_\alpha = \frac{\Phi_\alpha}{n_\alpha}, \qquad \text{where} \quad n_\alpha \in \mathbb{N}. \tag{4.83}$$

Upon this identification the bosonic part of the angular Hamiltonian (4.79) takes a form

$$\widetilde{\mathcal{I}}_0 = \frac{1}{2}\left(g + \sum_{\alpha=1}^{N-1} n_\alpha I_\alpha \right)^2, \qquad \text{with} \quad n_\alpha \in \mathbb{N}, \quad \Phi_\alpha \in [0, 2\pi), \tag{4.84}$$

but the bosonic generators $H_\alpha, S_\alpha, h_{\alpha\bar{\beta}}$, become locally defined, $\varphi_\alpha \in [0, 2\pi/m_\alpha)$, and fail to be constants of motion. To get the globally defined bosonic generators we have to take their relevant powers,

$$\widetilde{H}_\alpha := (H_\alpha)^{n_\alpha}, \quad \widetilde{K}_\alpha := (K_\alpha)^{n_\alpha}, \quad \widetilde{h}_{\alpha\bar{\beta}} := (h_{\alpha\bar{\beta}})^{n_\alpha n_\beta}. \tag{4.85}$$

as well as replace the fermionic generator $\Theta_{A\bar{\alpha}}$ by the following one

$$\widetilde{\Theta}_{A\bar{\alpha}} = (H_\alpha)^{n_\alpha - 1} \Theta_{A\bar{\alpha}}. \tag{4.86}$$

As a result, the dynamical (super)symmetry algebra becomes nonlinear deformation of $su(1, N|M)$

.

The angular Hamiltonian (4.84) define belong to the class the superintegrable generalizations of the conformal mechanics, and of the oscillator- and Coulomb-like systems on the N-dimensional Euclidian spaces [56]. As a particular cases, these set of systems includes the "charge-monopole" system, Smorodinsky-Winternitz system, as well as the rational Calogero models associated with Coxeter root system [2]. Thus, proposed systems can be considered as their $2M$ superconformal extensions.

Since the generators $Q_A, S_A, R_{A\bar{B}}$ remain unchanged upon above identification(as well as the expression of the angular Hamiltonian (4.39) via generators H, K, D), we conclude that listed generators form superconformal algebra $su(1,1|N)$ with central charge (4.64).

Finally, notice that in (4.84) the nonzero constant $g \neq 0$ appears, and the range of validity of the action variables is fixed to be $I_\alpha \in [0,\infty)$. As a result, standard free particle and conformal mechanics cannot be included in the proposed description, since for these systems we should choose $g = 0, I_\alpha \in [0,\infty)$. To exclude this constant we should replace the initial generators by the following ones

$$\mathcal{H} := H - \frac{g(g-2I)}{4K}, \qquad \mathcal{H}_\alpha := H_\alpha + \imath g\frac{K_\alpha}{2K}, \qquad \mathcal{Q}_A := Q_A - \imath g\frac{S_A}{2K}. \tag{4.87}$$

This deformation will further "non-linearize" the dynamical supersymmetry algebra $su(1,N|M)$.

4.5 OSCILLATOR- AND COULOMB-LIKE SYSTEMS

In the previous section we mentioned that the angular Hamiltonian (4.84) defines the superintegrable deformations of N-dimensional oscillator and Coulomb system [56], while in [17]

[2]To our best knowledge, the rational Calogero models are not yet constructed explicitly. However, we have at hand the spectra of quantum Calogero model and of its angular part [62]. Taking the (semi)classical limit of the spectrum of the angular Calogero model we can conclude that it is,indeed, of the form (4.84), see, e.g.[56]

the exhamples of such systems on noncompact projective space $\widetilde{\mathbb{CP}}^N$ playing the role of phase space were constructed. So, one can expect that on the phase superspace $\widetilde{\mathbb{CP}}^{N|M}$ one can construct the super-counterparts of that systems, which presumably, possess (deformed) $2M$-supersymmetric Poincaré supersymmetry. Below we examine this question and show that our claim is corrects in some particular cases.

4.5.1 OSCILLATOR-LIKE SYSTEMS

We define the supersymmetric oscillator-like system with the phase space $\widetilde{\mathbb{CP}}^{N|M}$ (equipped with the Poisson brackets (4.23)) by the Hamiltonian

$$H_{osc} = H + \omega^2 K, \tag{4.88}$$

where the generators H, K are given by (4.26). In canonical coordinates (4.73)it reads

$$H_{osc} = \frac{p_r^2}{2} + \frac{(g + \sum_{\alpha=1}^{N-1}\pi_\alpha + \sum_{A=1}^{M}\imath\bar{\chi}^A\chi^A)^2}{r^2} + \frac{\omega^2 r^2}{2}. \tag{4.89}$$

This system possesses the $u(N)$ symmetry given by the generators $h_{\alpha\bar\beta}$ defined in (4.27)(among them $N-1$ constants of motion π_α are functionally independent), the $U(M)$ R-symmetry given by the generators $R_{A\bar B}$ (4.29) as well as $N-1$ hidden symmetries given by the generators

$$M_{\alpha\beta} = (H_\alpha + \imath\omega K_\alpha)(H_\beta - \imath\omega K_\beta) = \frac{\bar{z}^\alpha\bar{z}^\beta}{A^2}(w^2+\omega^2)\ :\quad \{H_{osc}, M_{\alpha\beta}\} = 0, \tag{4.90}$$

The generators (4.90) and the $su(N)$ generators $h_{\alpha\bar\beta}$ form the following symmetry algebra

$$\{h_{\alpha\bar\beta}, M_{\gamma\delta}\} = \imath\left(M_{\alpha\delta}\delta_{\gamma\bar\beta} + M_{\gamma\alpha}\delta_{\delta\bar\beta}\right), \tag{4.91}$$

$$\{M_{\alpha\beta}, M_{\gamma\delta}\} = 0, \tag{4.92}$$

$$\{M_{\alpha\beta}, \overline{M}_{\gamma\delta}\} =$$
$$= \imath\left(4\omega^2 I h_{\alpha\bar\delta}h_{\beta\bar\gamma} - \frac{M_{\alpha\beta}\bar{M}_{\gamma\delta}}{h_{\alpha\bar\gamma}}\delta_{\alpha\bar\gamma} - \frac{M_{\alpha\beta}\bar{M}_{\gamma\delta}}{h_{\alpha\bar\delta}}\delta_{\alpha\bar\delta} - \frac{M_{\alpha\beta}\bar{M}_{\gamma\delta}}{h_{\beta\bar\gamma}}\delta_{\beta\bar\gamma} - \frac{M_{\alpha\beta}\bar{M}_{\gamma\delta}}{h_{\beta\bar\delta}}\delta_{\beta\bar\delta}\right), \tag{4.93}$$

with I given by (4.39) and summation over repeated indices is not assumed.

Besides, this system has a fermionic constants of motion $\Theta_{A\bar{\alpha}}$ defined in (4.28).Hence, it is superintegrable system in the sense of super-Liuville theorem, i.e. it has $2N-1$ bosonic and $2M$ fermionic, functionally independent, constants of motion [68]. Further generalization to the systems with angular Hamiltonian (4.84) is straightforward.

Let us show, that for the even $M = 2k$ this system possess the deformed $\mathcal{N} = 2k$ Poincaré supersymmetry, in the sense of papers written by E.Ivanov and S.Sidorov [44]. For this purpose we choose the following Ansatz for supercharges

$$\mathcal{Q}_A = Q_A + \omega C_{AB}\bar{S}_B, \tag{4.94}$$

with the constant matrix C_{AB} obeying the conditions

$$C_{AB} + C_{BA} = 0, \qquad C_{AB}\overline{C}_{BD} = -\delta_{A\bar{D}} \tag{4.95}$$

For sure, the condition (4.95) assumes that M is an even number, $M = 2k$.

Calculating Poisson brackets of the functions (4.94) we get

$$\{\mathcal{Q}_A, \bar{\mathcal{Q}}_B\} = H_{osc}\delta_{AB}, \qquad \{\mathcal{Q}_A, \mathcal{Q}_B\} = -\imath\omega\mathcal{G}_{AB}, \qquad \{\bar{\mathcal{Q}}_A, \bar{\mathcal{Q}}_B\} = \imath\omega\bar{\mathcal{G}}_{AB}, \tag{4.96}$$

where

$$\mathcal{G}_{AB} := C_{AC}R_{B\bar{C}} + C_{BC}R_{A\bar{C}}, \qquad \mathcal{G}_{\bar{A}\bar{B}} := \bar{\mathcal{G}}_{AB} = \bar{C}_{AC}R_{C\bar{B}} + \bar{C}_{BC}R_{C\bar{A}}, \tag{4.97}$$

and

$$\bar{\mathcal{G}}_{AB} = \bar{C}_{AC}\bar{C}_{DB}\mathcal{G}_{DC}. \tag{4.98}$$

Then we get that the algebra of generators $\mathcal{Q}_A$, $\mathcal{H}_{osc}$, $\mathcal{R}_A^B$ is closed indeed:

$$\{\mathcal{Q}_A, H_{osc}\} = \omega C_{AB}\mathcal{Q}_B, \qquad \{\mathcal{G}_{AB}, H_{osc}\} = 0, \tag{4.99}$$

$$\{\mathcal{Q}_A, \mathcal{G}_{BC}\} = \imath(C_{AB}\mathcal{Q}_C + C_{AC}\mathcal{Q}_B), \tag{4.100}$$

$$\{\mathcal{Q}_A, \bar{\mathcal{G}}_{BC}\} = -\imath(\bar{C}_{BD}\mathcal{Q}_D\delta_{A\bar{C}} + \bar{C}_{CD}\mathcal{Q}_D\delta_{A\bar{B}}). \tag{4.101}$$

$$\{\mathcal{G}_{AB}, \mathcal{G}_{CD}\} = \imath(C_{AD}\mathcal{G}_{BC} + C_{AC}\mathcal{G}_{BD} + C_{BD}\mathcal{G}_{AC} + C_{BC}\mathcal{G}_{AD}), \tag{4.102}$$

$$\{\mathcal{G}_{AB}, \bar{\mathcal{G}}_{CD}\} = \imath(\bar{C}_{DN}\delta_{AC} + \bar{C}_{CN}\delta_{AD})\mathcal{G}_{NB} + \imath(\bar{C}_{DN}\delta_{BC} + \bar{C}_{CN}\delta_{BD})\mathcal{G}_{NA}, \tag{4.103}$$

and surely,

$$\{\mathcal{Q}_A, H_{osc} + \frac{\imath\omega}{2}\sum_B \mathcal{G}_{BB}\} = 0. \tag{4.104}$$

Hence, for the $M = 2k$ the above oscillator-like system (4.88) possesses deformed $\mathcal{N} = 4k$ supersymmetry. In the particular case $M = 2$ the choose of the matrix C_{AB} is unique(up to unessential phase factor): $C_{AB} := \mathrm{e}^{\kappa}\varepsilon_{AB}$. In that case the above relations define the superalgebra $su(1|2)$-deformation of $\mathcal{N} = 4$ Poincaré supersymmetric mechanics studied in details in S.Sidorov [44, 45]. For the $k \geq 2$ the choice of matrices C_{AB} is not unique, and we get the family of deformed $\mathcal{N} = 4k$ Poincaré supersymmetric mechanics.

Let us present other deformed $\mathcal{N} = 2M$ Poincaré supersymmetryc systems whose bosonic part is different from those of (4.88) but nevertheless, has the oscillator potential.

For this purpose we choose another Ansatz for supercharges (in contrast with previous case M is not restricted to be even number)

$$\widetilde{\mathcal{Q}}_A = Q_A + \imath\omega S_A. \tag{4.105}$$

These supercharges generates the $su(1|M)$ superalgebra, and thus generalizes the systems considered in [44, 45] to arbitrary M,

$$\{\widetilde{\mathcal{Q}}_A, \bar{\widetilde{\mathcal{Q}}}_B\} = \mathcal{H}_{osc}\delta_{AB} - \omega\mathcal{R}^A_B, \tag{4.106}$$

$$\{\widetilde{\mathcal{Q}}_A, \widetilde{\mathcal{Q}}_B\} = 0, \tag{4.107}$$

$$\{\mathcal{R}_A{}^B, \mathcal{R}_C{}^D\} = \imath(\mathcal{R}_A{}^D\delta^B_C - \mathcal{R}_C{}^B\delta^D_A) \tag{4.108}$$

$$\{\widetilde{\mathcal{Q}}_A, \mathcal{R}^C_B\} = \imath\left(\frac{1}{M}\widetilde{\mathcal{Q}}_A\delta_{B\bar{C}} + \widetilde{\mathcal{Q}}_B\delta_{A\bar{C}}\right), \tag{4.109}$$

$$\{\widetilde{\mathcal{Q}}_A, \mathcal{H}_{osc}\} = \imath\omega\frac{2M-1}{M}\widetilde{\mathcal{Q}}_A, \tag{4.110}$$

where

$$\mathcal{H}_{osc} := H_{osc} - \omega(I + \frac{1}{M}\sum_C R_{C\bar{C}}), \qquad \mathcal{R}_A{}^B := R_{A\bar{B}} - \frac{1}{M}\delta^B_A\sum_C R_{C\bar{C}} \tag{4.111}$$

with I defined by (4.39). Hence, the Hamiltonian get the additional bosonic term proportional to the casimir of conformal group. In canonical coordinates (4.73) it reads

$$\mathcal{H}_{osc} = \frac{p_r^2}{2} + \frac{\mathcal{I}}{r^2} + \frac{\omega^2 r^2}{2} - \omega\left(\sqrt{2\mathcal{I}} + \frac{1}{M}(\bar{\chi}\chi)\right). \tag{4.112}$$

This Hamiltonian, seemingly, describes the oscillator-like systems specified by the presence of external magnetic field.

So, choosing $\widetilde{\mathbb{CP}}^{N|M}$ as a phase superspace, we can easily construct superintegrable oscillator-like systems which possess deformed $\mathcal{N} = 2M$ Poincar/'e supesymmetry.

4.5.2 COULOMB-LIKE SYSTEMS

Now, let us construct on the phase space $\widetilde{\mathbb{CP}}^{N|M}$ with the Poisson bracket relations (4.23), the Coulomb-like system given by the Hamiltonian

$$H_{Coul} = H + \frac{\gamma}{\sqrt{2K}}, \tag{4.113}$$

where the generators H, K are defined by (4.26).

The bosonic constants of motion of this system are given by the $u(N-1)$ symmetry generators $h_{\alpha\beta}$, and by the $N-1$ additional constants of motion

$$R_\alpha = H_\alpha + \imath\gamma\frac{K_\alpha}{I\sqrt{2K}} \quad : \quad \{H_{Coul}, R_\alpha\} = \{H_{Coul}, h_{\alpha\bar\beta}\} = 0, \tag{4.114}$$

where $H_\alpha, K_\alpha, \eta_{\alpha\bar\beta}$ are defined by (4.27). These generators form the algebra

$$\{R_\alpha, \bar{R}_{\bar\beta}\} = -\imath\delta_{\alpha\bar\beta}\left(H_{Coul} - \frac{\imath\gamma^2}{2I^2}\right) + \frac{\imath\gamma^2 h_{\alpha\bar\beta}}{2I^3}, \tag{4.115}$$

$$\{h_{\alpha\bar\beta}, R_\gamma\} = \imath\delta_{\gamma\bar\beta}R_\alpha, \tag{4.116}$$

$$\{R_\alpha, R_\beta\} = 0. \tag{4.117}$$

Besides, proposed system has $2M$ fermionic constants of motion given by $\Theta_{A\bar\alpha}$, and $u(M)$ R-symmetry given by $R_{A\bar B}$. Hence, it is superintegrable in the sense of super-Liuville theorem [68]. So, we constructed the maximally superintegrable Coulomb problem with dynamical $SU(1, N|M)$ superconformal symmetry which inherits all symmetries of initial bosonic system.

One can expect, that in analogy with oscillator-like system, our Coulomb-like system would possess (deformed) $\mathcal{N} = 2M$-super-Poincaré symmetry for $M = 2k$ and $\gamma > 1$. However, it is not a case.

Indeed, let us choose the following Ansatz for supercharges

$$\mathcal{Q}_A = Q_A + \sqrt{2\gamma}C_{AB}\frac{\bar{S}_B}{(2K)^{3/4}}, \tag{4.118}$$

with the constant matrix C_{AB} obeying the conditions (4.95), $M = 2k$ and $\gamma > 0$.

Calculating their Poisson brackets we find

$$\{\mathcal{Q}_A, \bar{\mathcal{Q}}_B\} = H_{Coul}\delta_{A\bar{B}} + \frac{3}{2}\frac{\sqrt{2\gamma}}{(2K)^{7/4}}\left(S_A\bar{C}_{BD}S_D + \bar{S}_BC_{AD}\bar{S}_D\right), \tag{4.119}$$

$$\{\mathcal{Q}_A, \mathcal{Q}_B\} = -\frac{\imath\sqrt{2\gamma}}{2(2K)^{3/4}}(C_{BD}\mathcal{R}^D_A + C_{AC}\mathcal{R}^D_B), \tag{4.120}$$

$$\{\mathcal{Q}_A, \mathcal{R}^C_B\} = -\imath\mathcal{Q}_B\delta_{A\bar{C}}, \tag{4.121}$$

where $\mathcal{R}^A_B$ is defined in (4.111).

Further calculating the Poisson brackets of $\mathcal{Q}_A$ with the generators appearing in the r.h.s. of the above expressions we get that the superalgebra is not closed. For example,

$$\{\mathcal{Q}_A, H_{Coul}\} = \frac{3\gamma}{(2K)^{3/2}}S_A + \frac{\sqrt{2\gamma}}{(2K)^{3/4}}C_{AB}\left(\bar{\mathcal{Q}}_B - \frac{3}{4K}\bar{S}_BD\right). \tag{4.122}$$

Hence, proposed supercharges do not yield closed deformation of $\mathcal{N} = 2M$-super-Poincaré algebra.

Let us choose another Ansatz for supercharges (as above we assume that $\gamma > 0$)

$$\widetilde{\mathcal{Q}}_A = Q_A + \imath\sqrt{2\gamma}e^{\imath\frac{\pi}{2}}\frac{S_A}{(2K)^{3/4}}, \tag{4.123}$$

which yields

$$\{\widetilde{\mathcal{Q}}_A, \bar{\widetilde{\mathcal{Q}}}_B\} = \mathcal{H}_{Coul}\delta_{A\bar{B}} + \frac{\sqrt{2\gamma}}{2(2K)^{3/4}}\mathcal{R}^B_A, \tag{4.124}$$

$$\{\widetilde{\mathcal{Q}}_A, \mathcal{R}^C_B\} = \imath\left(\frac{1}{M}\widetilde{\mathcal{Q}}_A\delta_{B\bar{C}} - \widetilde{\mathcal{Q}}_B\delta_{A\bar{C}}\right), \tag{4.125}$$

$$\{\widetilde{\mathcal{Q}}_A, \widetilde{\mathcal{Q}}_B\} = \{\bar{\widetilde{\mathcal{Q}}}_A, \bar{\widetilde{\mathcal{Q}}}_B\} = 0, \tag{4.126}$$

where

$$\mathcal{H}_{Coul} = H_{Coul} - \frac{\sqrt{2\gamma}}{(2K)^{3/4}}\left(I - \frac{1}{2M}\sum_C R_{C\bar{C}}\right), \tag{4.127}$$

with I and $\mathcal{R}^A_B$ are defined, respectively, in (4.79) and (4.111). In canonical coordinates (4.73) this Hamiltonian reads

$$\mathcal{H}_{Coul} = \frac{p_r}{2} + \frac{\mathcal{I}}{r^2} + \frac{\gamma}{r} - \frac{\sqrt{2\gamma}}{r^{3/2}}\left(g + \sum_\alpha \pi_\alpha + \frac{2M-1}{2M}(\bar{\chi}\chi)\right). \tag{4.128}$$

However, one can easily check that proposed supercharges do not yield closed deformation of Poincaré superalgebra as well, e.g.

$$\{\widetilde{\mathcal{Q}}_A, \frac{\mathcal{R}_B^C}{(2K)^{3/4}}\} = \frac{\imath}{(2K)^{3/4}}\left(\frac{1}{M}\widetilde{\mathcal{Q}}_A\delta_{B\bar{C}} - \widetilde{\mathcal{Q}}_B\delta_{A\bar{C}}\right) + \frac{3}{2}\frac{S_A}{(2K)^{7/4}}\mathcal{R}_B^C \tag{4.129}$$

So, proposed superextensions of Coulomb-like systems, being well-defined from the viewpoint of superintegrability, do not possess neither $\mathcal{N} = 2M$ supersymmetry, no its deformation. The $su(1, N|M)$ superalgebra plays the role of dynamical algebra of that systems.

4.6 FUBINI-STUDY-LIKE KÄHLER STRUCTURE

The above considered super-Kahler structure is obviously the higher-dimensional superanalog of the Klein model of Lobachevsky space. On the other hand, Lobachevsky space has other common parametrization as well, which is known as Poincaré disc. The higher-dimensional generalization of Poincaré disc parameterizing the noncompact complex projective space is quite similar to the Fubini-Study structure for $\mathbb{CP}^N$, and is defined by the Kähler potential

$$\mathcal{K} = -g\log(1 - \sum_{a=1}^{N} z^a\bar{z}^a). \tag{4.130}$$

For the obtaining of the superanalog of this potential from $\mathbb{C}^{1,N|M}$, one should transit from the matrix (4.7) to the diagonal matrix $\gamma_{a\bar{b}} = \mathrm{diag}(1, -1, \ldots, -1)$, which can be done by the transformation

$$v^0 \to \frac{v^0 + v^N}{\sqrt{2}}, \qquad v^N \to \frac{v^0 - v^N}{\imath\sqrt{2}}. \tag{4.131}$$

In these terms the Poisson brackets will again be given by the are given by the relations (4.8), but with $\gamma_{a\bar{b}} = diag(1, -1, \ldots, -1)$.

On the reduced phase space (4.131) corresponds to the transformation

$$w \to \imath\frac{z^N - 1}{z^N + 1}, \qquad z^\alpha \to \sqrt{2}\frac{z^\alpha}{z^N + 1}, \qquad \theta^A \to \sqrt{2}\frac{\theta^A}{z^N + 1}. \tag{4.132}$$

Doing so we will get the Fubini-Studi-like Kähler potential

$$\mathcal{K} = -g \log(1 - z^c \bar{z}^c + \imath \theta^C \bar{\theta}^C), \tag{4.133}$$

which defines the following Kähler structure

$$\begin{aligned}\Omega = \frac{\imath}{g}\Bigg[&\left(\frac{g\delta_{a\bar{b}}}{\tilde{A}} + \frac{\bar{z}^a z^b}{\tilde{A}^2}\right) dz^a \wedge d\bar{z}^b + \frac{\imath\bar{\theta}^A z^a}{\tilde{A}^2} d\theta^A \wedge d\bar{z}^a \\ &- \frac{\imath \bar{z}^a \theta^A}{\tilde{A}^2} dz^a \wedge d\bar{\theta}^A - \left(\frac{g\delta_{A\bar{B}}}{\tilde{A}} + \frac{\bar{\theta}^A \theta^B}{\tilde{A}^2}\right) d\theta^A \wedge d\bar{\theta}^B\Bigg],\end{aligned} \tag{4.134}$$

where we have used a similar notation as in (4.18)

$$\tilde{A} := \frac{1 - z^c \bar{z}^c + \imath \theta^C \bar{\theta}^C}{g}. \tag{4.135}$$

The respective Poissoon brackets read:

$$\{z^a, \bar{z}^b\} = \frac{\delta^{a\bar{b}} - z^a \bar{z}^b}{\tilde{A}}, \qquad \{z^a, \bar{\theta}^A\} = \frac{z^a \bar{\theta}^A}{\tilde{A}}, \qquad \{\theta^A, \bar{\theta}^B\} = \frac{\delta^{A\bar{B}} + \theta^A \bar{\theta}^B}{\tilde{A}}. \tag{4.136}$$

Now let us repeat the same procedure of introduction of canonical coordinates, but now taking the symplectic/Kḧler one form associated with the Kähler potential (4.133), i.e. the one that define "Fubini-Study"-like metric. Then, as before, one needs to identify it with the canonical one, and this canonical coordinates will play the role of "Cartesian" coordinates instead of the "spherical" ones discussed above.

$$\begin{aligned}\tilde{\mathcal{A}} &= -\frac{g}{2} \frac{\imath(\bar{z}^a dz^a - z^a d\bar{z}^a) + \theta^A d\bar{\theta}^A + \bar{\theta}^A d\theta^A}{1 - z^c \bar{z}^c + \imath \theta^C \bar{\theta}^C} \\ &:= p_a d\varphi_a + \frac{1}{2}\chi^A d\bar{\chi}^A + \frac{1}{2}\bar{\chi}^A d\chi^A.\end{aligned} \tag{4.137}$$

It leads us to

$$z^a = \sqrt{\frac{p_a}{g + p - \imath \chi^C \bar{\chi}^C}} e^{\imath \varphi_a}, \qquad \theta^A = \frac{\sqrt{2}}{r}\chi^A, \tag{4.138}$$

with

$$p = \sum_a p_a. \tag{4.139}$$

And vice versa,

$$p_a = \frac{z^a \bar{z}^a}{\tilde{A}}, \qquad \varphi_a = \arg(z^a), \qquad \chi^A = \frac{\theta^A}{\sqrt{\tilde{A}}}, \tag{4.140}$$

where $\tilde{A}$ is defined by (4.135).

These coordinates are related with (4.73) as follows

$$p_N = \frac{1}{4}\left(p_r^2 + \left(r - \frac{\sqrt{2\mathcal{I}}}{r}\right)^2\right), \tag{4.141}$$

$$\varphi_N = \arctan\left(\frac{2xy}{(x-y)(x+y)}\right), \tag{4.142}$$

$$p_\alpha = \pi_\alpha, \quad \chi^A = \chi^A, \tag{4.143}$$

where

$$x = 1 - \frac{p_r^2}{r^2} - \frac{2\mathcal{I}}{r^4}, \quad y = \frac{p_r}{r}. \tag{4.144}$$

Finally, let us draw readers attention to the complete similarity of the bosonic part of (4.138) with the equations mapping parameterizing compactified Ruijsenaars-Schneider model with excluded centre of mass to the complex projective (phase) space $\mathbb{CP}^N$. This prompt us, at first, to construct the conformal-invariant analog of that model by replacing the complex projective space by its noncompact analog $\widetilde{\mathbb{CP}}^N$. Then one can try to construct its $su(1,N|M)$-supeconformal extension by further replacement of $\widetilde{\mathbb{CP}}^N$ by $\widetilde{\mathbb{CP}}^{N|M}$.

4.7 5-GRADING AND COSET CONSTRUCTION

The key point of our consideration in the previous Sections was the reduction of the complex Euclidian superspace $\mathbb{C}^{1,N|N}$ to the Kähler phase superspace with the non-compact symmetry superalgebra algebra $\mathcal{G} = su(1,N|M)$ containing conformal algebra $su(1,1)$ as the subalgebra subgroup of the symmetry algebra $\mathcal{G}$, and admitting 5-graded decompositions with respect to dilatation generator D.

However, it is a well known fact [46, 47] that every simple Lie algebra $\mathcal{F}$ (except for sl_2) admits 5-graded decompositions with respect to a suitable generator $D \in su(1,1) \in \mathcal{F}$,

$$\mathcal{F} = \mathfrak{f}_{-2} \oplus \mathfrak{f}_{-1} \oplus \mathfrak{f}_0 \oplus \mathfrak{f}_{+1} \oplus \mathfrak{f}_{+2} \qquad \text{with} \qquad [\mathfrak{f}_i, \mathfrak{f}_j] \subseteq \mathfrak{f}_{i+j} \quad \text{for } i,j \in \{-2,-1,0,1,2\} \tag{4.145}$$

($\mathfrak{f}_i = 0$ for $|i| > 2$ understood). Compatibility with the 5-grading requires this real form to be non-compact. Therefore, (H, D, K) generate an $su(1,1)$ subalgebra of $\mathcal{F}$. Different real forms of $\mathcal{F}$ and $\mathcal{H}$ give rise to different non-compact quaternionic symmetric spaces W [46, 47],

$$W = \frac{F}{H \times \mathrm{SU}(1,1)}, \tag{4.146}$$

where F, H and SU(1,1) are the (simply-connected) groups generated by $\mathcal{F}$, $\mathcal{H}$ and $su(1,1)$, respectively.

One may enlarge the coset by reducing the stability group [48] from $H \times \mathrm{SU}(1,1)$ to $H \times \mathfrak{B}_{\mathrm{SU}(1,1)}$, where $\mathfrak{B}_{\mathrm{SU}(1,1)}$ denotes the positive Borel subgroup of SU(1,1), whose algebra $\mathfrak{b}_{su(1,1)}$ is generated by (D, K). In other words, we keep H in the numerator and consider the coset

$$\mathcal{W} = \frac{F}{H \times \mathfrak{B}_{\mathrm{SU}(1,1)}}. \tag{4.147}$$

The elements of $\mathcal{W}$ can be parametrized as follows,

$$g = e^{t\left(H + \omega^2 K\right)} e^{u(t)\cdot G_{-1}} e^{v(t)\cdot G_1}, \tag{4.148}$$

where we employed a $\cdot$ notation to suppress the summation over A. The parameter ω represents some freedom in the parametrization of $\mathcal{W}$. realization of the group $\mathcal{F}$ by the left multiplications on the coset $\mathcal{W}$ (4.148) will give rize to a proper realization of the symmetry in terms of basic fields $u(t), v(t)$, while the dynamic equations can be obtained by imposing the proper constraints on the Cartan forms [48]. Defining the Cartan forms in the standard way,

$$g^{-1}dg = \omega_{-2} H + \omega_0 D + \omega_2 K + \omega_{-1} \cdot G_{-1} + \omega_1 \cdot G_1 + \sum_s \omega_h^s \, h_s, \tag{4.149}$$

one can check that the constraints

$$\omega_{-1} = 0 \tag{4.150}$$

firstly are invariant under the whole group F, realized by left multiplication in the coset $\mathcal{W}$ (4.147), and secondly express the Goldstone fields $v(t)$ through the Goldstone fields $u(t)$ and their time derivatives in a covariant fashion. Thus means, that the fields $v(t)$ acquire meaning of the momenta for the fields $u(t)$ after passing to the Hamiltonian formalism.After imposing the constraints (4.150) we have a realization of the F transformations on the time t and the d coordinates $u_A(t)$.

Finally, one can impose the additional invariant constraints

$$\omega_1 = 0, \tag{4.151}$$

which produces a system of second-order differential equations for the variables $u_A(t)$. These are the equations of motion. Hence, with every simple Lie algebra $\mathcal{F}$ one may associate a system of dynamical equations in d variables which is invariant under some non-compact real form of the group F.

4.8 CONCLUSION

In this chapter we suggest the $su(1.N|M)$ superconformal mechanics formulated in terms of phase superspace given by the noncompact analogue of complex projective superspace. We parametrized this phase space by the specific coordinates allowing us to interpret it as a higher-dimensional superanalogue of the Lobachevsky plane parametrized by lower half-plane (Klein model). Then we introduced the canonical coordinates corresponding to the known separation of the "radial" and "angular" parts of (super)conformal mechanics. Relating the "angular" coordinates with action-angle variables, we demonstrated that the proposed scheme allows us to construct the $su(1.N|M)$ supeconformal extensions of wide class of superintegrable systems. We also proposed the superintegrable oscillator- and Coulomb-like systems with a $su(1.N|M)$ dynamical superalgebra and found that oscillatorlike systems admit deformed $\mathcal{N} = 2M$ Poincaré supersymmetry, in contrast with Coulomb-like ones.

Chapter 5

Discussion

As a summary, in this chapter we bring the main results of the thesis.

In the *first chapter* we have discussed some basics of Hamiltonian formalism, the geometry of integrability, specially we have considered the use of the Kähler manifold regarded as a phase space of Hamiltonian systems. Some examples of maximally integrable systems and maximally symmetric Kähler (phase) spaces have been illustrated.

In the *second chapter* we formulated the Euler top as a system with phase space $\mathbb{CP}^1$, i.e. as one-dimensional system. Then we proposed the procedure of $\mathcal{N} = 2k$ *á priori* integrable supersymmetrization of a generic one-dimensional systems which provides the family of $\mathcal{N}$-supersymmetric extensions depending on $\mathcal{N}/2$ arbitrary real functions. Thus, we gave the $\mathcal{N} = 2k$ supersymmetric extensions of the Euler top as well.

A few more comments on this topic are worth to be mentioned. One may ask whether it is possible to construct the family of supersymmetric extensions of the Lagrange and Kowalewski tops (see, e.g.,[69]) which are parameterized by arbitrary functions?

Here we present some preliminary remarks on this issue. The phase spaces of Lagrange and Kowalewski tops could be identified with cotangent bundle of complex projective plane. This supermanifold can be equipped with three symplectic (and complex) structures, parameterized by the coordinates $u^A = (z, \pi)$,

$$\omega_1 = d\pi \wedge dz + d\bar{\pi} \wedge d\bar{z} \tag{5.1}$$

$$\omega_2 = \imath d\pi \wedge dz - \imath d\bar{\pi} \wedge d\bar{z}, \tag{5.2}$$

$$\omega_3 = \imath \frac{\partial^2 \tilde{K}}{\partial u^A \partial \bar{u}^B} du^A \wedge d\bar{u}^B, \tag{5.3}$$

$$\tilde{K} = K(z, \bar{z}) + F(g^{-1}\bar{\pi}\pi), \tag{5.4}$$

where $K(z, \bar{z})$ and $g(z, \bar{z})$ are given by (2.10) and(2.9) respectively, while $F(x)$ is the real function obeying condition $F'(0) \neq 0$. Within appropriate choice of the function $F(x)$ these symplectic structures provide the manifold $T^*\mathbb{CP}^1$ with hyper-Kähler structure [70]. Formulating Lagrange and Kowalewski tops in terms of symplectic structures (5.2), we can try to construct their conventional $\mathcal{N} = 2, 4$ supersymmetric extensions, extending these simplectic structure by fermionic variables associated with dz. However, we are expecting that using the symplectic structure (5.4) will be more useful for the construction of the supersymmetric extensions of Lagrange and Kowalewski tops.

In the *third chapter* we have shown that the superintegrable generalizations of conformal mechanics, oscillator and Coulomb systems can be naturally described in terms of the non-compact complex projective space considered as a phase space. This observation yields some interesting directions for further studies.

For example, performing the transformation to the higher-dimensional Poincare model via (3.23), we expect to present the considered models in the Ruijsenaars-Schneider-like form and in this way to find, some superinegrable cases of the Ruijsenaars-Schneider systems, as well as their supersymmetric/superconformal extensions.

Another one is describing the superintegrable deformations of the free particle on the spheres/hyperboloids, and the spherical/hyperbolic oscillators, in a similar way. For this purpose we expect to consider the "κ-deformation" of the Kähler structure of the Klein model, in the spirit of the so-called "κ-deformation approach" developed in [53].

As well as, we are going to undertake the construction of spin-extensions for the aforementioned models, opting for the noncompact analogs of complex Grassmannians as phase spaces.

In *Chapter 4* we suggested to construct the $su(1,N|M)$-superconformal mechanics formulating them on phase superspace given by the non-compact analog of complex projective superspace $\mathbb{CP}^{N|M}$. The $su(1,N|M)$ symmetry generators were defined there as a Killing potentials of $\mathbb{CP}^{N|M}$. We parameterized this phase space by the specific coordinates allowing to interpret it as a higher-dimensional super-analog of the Lobachevsky plane parameterized by lower half-plane (Klein model). Then we transited to the canonical coordinates corresponding to the known separation of the "radial" and "angular" parts of (super)conformal mechanics. Relating the "angular" coordinates with action-angle variables we demonstrated that proposed scheme allows to construct the $su(1,N|M)$ supeconformal extensions of wide class of superintegrable systems. We also proposed the superintegrable oscillator- and Coulomb- like systems with a $su(1,N|M)$ dynamical superalgebra, and found that oscillator-like systems admit deformed $\mathcal{N}=2M$ Poincaré supersymmetry, in contrast with Coulomb-like ones.

In fact, proposed scheme demonstrated the effectiveness of the supersymmetrization via formulation of the initial systems in terms of Kähler phase space and further generalisation of the latter ones. In order to relate considered systems with the conventional ones (with Euclidean configuration spaces) , we restricted ourselves by the non-compact complex projective superspace. So, we are sure that applying the same approach to the conventional (compact) complex projective spaces we can find many new integrable systems as well and construct their unpredictable extended supersymmetric extensions.

Proposed scheme could obviously be extended to the systems on complex Grassmanians $Gr_{N.K}(\mathbb{C})$ (and on their noncompact analogs). In particular, we expect to find, in this way, the $\mathcal{N}$-supersymmetric extensions of compactified spin-Ruijsenaars-Schneider models. Moreover, it seems to be straightforward task to apply proposed approach to the systems with generic $U(N)$-invariant Kähler phase spaces locally defined by the Kähler potential $\mathcal{K}(\sum_{a=1}^{N} z^a\bar{z}^a)$. We expect that it can be done in terms of Kähler phase superspace locally defined by the potential

$$\widetilde{\mathcal{K}} = \mathcal{K}\left(\sum_{a=1}^{N} z^a\bar{z}^a + \imath\sum_{A=1}^{M}\eta^A\bar{\eta}^A\right). \tag{5.5}$$

Finally, notice that our configuration is not coherent with the superfield approach to supersymmetric mechanics, since considered phase superspace is not associated with external algebra of initial bosonic manifolds.

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